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THE APPORTIONMENT PROBLEM AND ITS APPLICATION IN DETERMINING POLITICAL REPRESENTATION

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Abstract

In this paper we give an overall view of the apportionment problem and its applications and, in particular, we discuss the determination of political representation. We comment on the methods currently being used in Greece and in the United States of America for allocating representatives to their geographical regions and we offer alternatives which may be considered for future apportionment. (JEL C15)

1. Introduction

There is a large class of real life problems related to fairness in division and apportionment. For example, how one should allocate seats propostionally to party vote totals? In manpower planning the allocation of jobs in proportion to certain characteristics of the labor pool can be a problem. Service facilities —court, judges, or hospitals— may need to be allotted to areas in proportion to the number of people to be served. Any problem in which objects are to be distributed in non-negative integers proportionally to some numerical criterion belongs to this class and, is clearly connected to the statistical problem of making rounded percentages add up toy 100%.

A perfectly fair division is impossible to achieve due to the indivisibility of the objects. In this paper, after an overall view of the apportionment problem we take a closer and more detailed look at one of its applications, namely the determination of political representation. We comment on the Hamilton method, being used in Greece for the allocation of seats of the Greek Parliament to the 56 nomos (districts). We do the same for the Hill method, currently being used in the United States to determine the allocation of the seats of the House of

Representatives to the 50 States. We point out the weak and strong points of these methods and we examine alternatives which may be under consideration in the future.

2. History of the Apportionment Problem. Notation and Preliminaries

The apportionment problem arises every time we are required to round fractions so that their sum is maintained at some given constant value. It appears in many situations, for example, in allocating seats of a legislature according to the populations of districts or to party votes, in assigning faculty to colleges or departments according to the number of students attending these colleges or departments, in allotting service facilities (courts, judges, or hospitals) to areas in proportion to the number of people to be served; in assigning buses to different routes to best meet the expected demand; in reporting statistical findings where we may wish to round percentages while we maintain the sum at 100%.

In this section we first describe the problem of fair representation. What is a fair way to determine political representation in democratic institutions?

In order to practice the ideal of one-man, one-vote, the democratic nations of the world have been using either of the following systems or a combination of them.

1. The federal system, where the unit of representation is regional, that allocates seats to states or provinces according to their population.

2. The proportional representation system, where the unit is political, that gives seats to a party according to its vote.

For example, the United States has a federal system, Israel has a proportional representation system, while Germany, Switzerland and Greece have a combination of the two systems.

In all of the systems the aim is that no individual should have a greater voice than another: a district should receive a number of representatives in proportion to its population or a party in proportion to its total vote.

Although proportionality seems to be the solution, it cannot be met in practice. The problem that arises is what to do about the fractions. A representative cannot be cut in pieces! In the United States, there have been many debates over the choice of method to solve the apportionment problem. Beginning at the

Constitutional Convention in 1787, the issue came up again in 1791, after the first census was reported, and resurfaces every ten years with the completion of every cencus. The most recent debate was on March 31, 1992 (New York Times, April 1, 1992) when the Supreme Court upheld the constitutionality of the current method of apportionment (Hill's method or method of Equal Proportions) denying to grant the state of Montana an extra seat.

Next, we define the quota, the fair share and the quotient of a state (or district). The *quota* q_i for each state i is found by dividing the states' population p_i by the total population P and then multiplying by the total number h of seats to be apportioned, i.e., $q_i = \frac{p_i h}{P}$.

The *quotient* Q_i of a state is computed by dividing the state's population by a divisor d, where the value of d depends on the method to be followed for apportioning seats into states. Specifically, d must be selected in such a way, so that appropriate rounding of the resulting quotients produce numbers a_i that sum to the given house size h.

In many countries, there are prescribed "floors" and "ceilings" on the states permissible allotments. For every state i, $f_i \leq a_i \leq c_i$ where f_i and c_i are the minimum and maximum number of seats that the i^{th} state may receive, and a_i is the number of seats that the i^{th} state receives. The *fair share* s^i of the i^{th} state is defined as $s_i = \text{med}\{f_i, \lambda q_i, c_i\}$, where q_i is the quota of the i^{th} state and λ is such that $\Sigma s_i = h$. Clearly, if there are no floors and ceilings then $s_i \equiv q_i$ for every state i.

A state should receive its fair share but, most of the time, the fair share is not an integer number. The matter has concerned many mathematicians and noted statesmen. This fact attests both to the complexity of the problem and to its profound political consequences. For example, in U.S.A., some of the greatest disputes over method have been over the allocation of a single seat. In Greece, on the other hand, the primary controversies are over the political makeup of each of the districts representatives, rather than over the method of allocation of seats to each district. That is, there is more attention paid to who in particular (which political party) is going to get the seats of the parliament than to the number of seats allotted to each district.

In the next section, we refer to the most popular of the methods of apportionment, existing around the world.

3. Methods of Apportionment

Following are some of the most acceptable methods for apportioning a specified number h of seats to states or electoral districts:

Alexander Hamilton finds the fair share of each state and gives to each of them the whole number contained in the fair share, assigning any seats which are as yet unapportioned to the states having the largest remainders.

William Lowndes gives also the whole number contained in the fair share, but assigned the unapportioned seats to the states having the largest adjusted remainder, where

$$\text{adjusted remainder:} = \frac{\text{remainder}}{\text{whole number in the fair share}}$$

Thomas Jefferson, John Quincy Adams, James Deam, Joseph Hill and Daniel Webster introduced methods which belong to another wide category called *divisor methods.* The idea is to choose some ratio of population to representatives (called divisor), and then divide this divisor into the population of states to obtain quotiens. The quotients are rounded up or down to a neighboring whole number according to a rule that depends on the particular method. The whole numbers so obtained must sum to the required number of seats (size of the House). The divisor is adjusted upward or downward if the sum is too large or too small respectively. Note that there is an interval of workable values of the divisor, rather than a unique solution.

Here is how these methods work:

Jefferson finds such a divisor so that the whole numbers contained in the quotients of states sum to h, and gives to each state its whole number. In some countries (for example, Austria), Jefferson's method is known as D'Hondt's method.

Adams finds such a divisor so that the smallest whole numbers containing the quotients o the states sum to h and gives to each state its whole number.

Webster finds such a divisor so that the whole numbers nearest to the quotients of states sum up to h, and then gives to each state its whole number.

Before we introduce the apportionment methods of Deam and Hill we need the following definition:

The *constituency size* of a state is defined to be the number of persons per representative in this state.

Dean finds a divisor d so that the whole numbers, which bring the average constituencies of the states closer to d sum up to h, and then gives to each state its whole number. In other words, Dean choose a common divisor d and gives to each state that number of seats which brings its number of inhabitants per representative closer to d. In practice, Dean's apportionment is obtained by dividing each state's population by d, and then giving the state either its quotient rounded-up or rounded-down, depending on whether or not the quotient exceeds the harmonic mean of the two choices. (The harmonic mean of two numbers is their product divides by their average). The sum of the given whole numbers must be h, otherwise d need to be adjusted.

Hill gives to each state a number of seats so that no transfer of any one seat can reduce the percentage difference in representation between those states. To do this, Hill introduces a divisor d and gives to each state that number of seats which brings its constituency size closer to d in relative term. In practice, Hill's apportionment is obtained by dividing each state's population by d, and then giving to each state either its quotient rounded-up or rounded-down, depending on whether or not the quotient exceeds the geometric mean of these two choices. (The geometric mean of two numbers is the square root of their product). The divisor d needs to be adjusted if the assigned number of seats do not sum to h. In Section 4, we will present the exact steps being taken in the United States in computing apportionment.

There are countless more methods of apportionment which are either Hamilton-type or belong to the category of the divisor methods. But which method should we be using? Are there any other methods to which we should pay attention? What constitutes fairness in the problem of apportionment? It seems that different definitions of "fairness" indicate different apportionment methods to be appropriate. Also for different distributions of the population we may need to use different apportionment methods.

4. Choosing a Fair Method of Apportionment

An apportionment problem is given by a vector of populations $\vec{p} = (p_1, ..., p_n)$ of n states, a total number of seats h to be distributed among the n states, a vector of floors $\vec{f} = (f_1, ..., f_n)$ and a vector of ceilings $\vec{c} = (c_1, ..., c_n)$. An apportionment of h is a vector $\vec{a} = (a_1, ..., a_n)$ where $\forall i = 1, ..., n$, a_i is nonnegative integer, $f_i \leq a_i \leq c_i$ and $\sum_{i=1}^{n} a_i = h$. Given an apportionment problem $(\vec{p}, h, \vec{f}, \vec{c})$, the objective is to find a vector $\vec{a}$ of apportionments that most closely approximates the vector of fair share $\vec{s} = \{s_1, ..., s_n\}$. But should we measure the approximation? And, is the optimization of this measure enough to determine the most fair method?

Many "paradoxes" that a fair method should avoid, have arisen in the history of the United States. Some well known paradoxes are:

(i) The "Alabama paradox", where a state being allotted y seats out of a total of x, receives y-1 when the total becomes x+1.

(ii) The "Population paradox", where state A has been given x seats, state B has been given y seats and a year later, although state A was growing faster than state B, state A loses a seat to state B.

(iii) The "New States paradox", where when a new state enters the Union with its fair share, state A loses a seat to state B.

An apportionment method M is *house monotone* if whenever $\vec{a}$ is an M-apportionment for $(\vec{f}, \vec{c}, \vec{p}, h)$ then there is some M-apportionment $\vec{a}\,'$ for (f, $\vec{c}, \vec{p}$, h+1) such that $a_i' \geq a_i$ for all i. Obviously a house monotone apportionment method avoids the Alabama paradox.

An apportionment method is *population monotone* when no state can lose a seat if only its population increases, and it is *quota monotone* when no state can lose a seat whenever its quota increases.

A fair apportionment method should stay *within fair share* (i.e. for every state i, $|s_i - a_i| < 1$) or at least *near fair share* (i.e., no transfer of a seat between two states brings both of them nearer their fair shares). It should not favor systematically large states at the expense of the small nor the small at the expence of the large, that is, it should be unbiased. An *unbiased* method is one that sometimes favors the large states and sometimes favors the small, but over many problems these advantages balance out.

Some of the elementary properties that a method should enjoy are homogeneity, symmetry, exactness and uniformity:

An apportionment method M is *homogeneous* if the M-apportionments for $\vec{p}$ and h are the same as the M-apportionments of $\lambda\vec{p}$ and h, for any positive rational number λ.

An apportionment method M is *symmetric* if permuting the populations to obtain a "new" problem only results in apportionments that are permuted in the same way.

An apportionment method M is *exact* if whenever an apportionment $\vec{a}$ is proportional to $\vec{p}$, satisfies the constraints and $\sum_{i=1}^{n} a_i = h$, then $\vec{a}$ is the unique M-apportionment for $\vec{p}$.

An apportionment method M is *Uniform* if for every t, $2 \leq t \leq n$, $(a_1, a_2, \ldots, a_n) \in M((p_1, \ldots, p_n), h)$ implies $(a_1, \ldots, a_t) \in M\left((p_1, \ldots, p_t), \sum_{i=1}^{t} a_i\right)$ and if also $(b_1, \ldots, b_t) \in M\left((p_1, \ldots, p_t), \sum_{i=1}^{t} a_i\right)$ then $(b_1, \ldots, b_t, a_{t+1}, \ldots, a_n) \in M((p_1, \ldots, p_n), h)$.

Remark (Balinski and Young, 1982)

All the methods to which we refer are homogeneous, symmetric and exact. A method M is uniform and exact if and only if it is a divisor method. A method is population monotone if and only if it is a divisor method. Every divisor method is house monotone and has particular solutions that avoid the Alabama paradox.

There exists no uniform and symmetric method that satisfies fair share. A method M is uniform and near the fair share if and only if it is the Webster method. Empirical observation and theory show that Webster's is the unique unbiased divisor method, i.e. Webster's method does not have a systematic tendency to favor the large states at the expense of the smaller ones or the small at the expense of the larger.

It is clear from the above remark that none of the methods we have mentioned has all the desirable properties. It is evident though, that most of the properties hold for the divisor methods.

Any monotone increasing function d: $Z \rightarrow R$ with $a \leq d(a) \leq a+1$ for every $a \in Z$ is said to be a divisor criterion.

Different divisor methods are based on different divisor criteria d. Difine

$$\left[\frac{p_i}{x}\right]_d := \begin{cases} k & \text{if } k < \frac{p_i}{x} < d(k) \quad \text{or } \frac{p_i}{x} = d(k) \text{ and k odd,} \\ k=1 & \text{if } d(k) < \frac{p_i}{x} \leq k+1 \quad \text{or } \frac{p_i}{x} = d(k) \text{ and k even,} \end{cases}$$

where $k \in Z$.

Then, a divisor method based on d is defined, for each problem of apportionment $(\vec{f}, \vec{c}, \vec{p}, h)$, to be that apportionment $\vec{a}$ that satisfies for all i,

$$a_i = \text{med}\left\{ f_i, \left[\frac{p_i}{x}\right]_d, c_i \right\}, \text{ where x is chosen so that } \sum_{i=1}^{n} a_i = h. \ \square$$

We introduce some divisor methods. The divisor criteria used by those methods, for every $k \in Z$, are:

Adams	$:d(k) = k$	
Dean	$:d(k) = \dfrac{k\,(k+1)}{k+\frac{1}{2}}$	(Harmonic mean of k and k+1),
Hill	$:d(k) = \sqrt{k\,(k+1)}$	(Geometric mean of k and k+1),
Webster	$:d(k) = k + 1/2$	(Arithmetic mean of k and k+1),
Jefferson	$:d(k) = k + 1.$	

In this point we introduce the divisor criteria for some of the divisor methods we described in previous papers (see papers by B. D. Athanasopoulos).

1-Stationary : $d(k) = \frac{1}{2}$ if $k \neq 0$, $k \in Z$, $d(0) \in [0,1]$,

2-Stationary: $d(k) = \frac{1}{2}$ if $k \neq 0, 1$, $k \in Z$, $d(0) \in [0,1]$, $d(1) \in [0,1]$.

After applying an apportionment method, between any two states there will practically be a certain inequality which gives one of the states a slight advantage over the other. A transfer of one representative from the more favored state to the less favored should be made if the transfer will cause a decrease in the amount of the inequality between the two states. An apportionment method is called *stable* if no transfer is justified.

The question now that arises is what the measure of inequality should be. The U.S. Constitution expresses two ideals that are suitable for measuring inequalities: Firstly, every respresentative should have as nearly as possible the same number of constituents, i.e. the sizes of congressional districts should be as nearly as possible equal. Secondly, every inhabitant, no matter in what state he lives, should have as nearly as possible the same representation in the House of Representatives. Note that the size of congressional district in the ith state is given by the ratio $\frac{p_i}{a_i}$, where p_i is the population of the ith state and a_i is the number of seats apportioned to the ith state. On the other hand, the number of representatives per person in the ith state is given by the ratio $\frac{a_i}{p_i}$. Other possible measures of the inequality between two states i and j are:

$$a_j - a_i\,(p_j/p_i),\ a_j\,(p_i/p_j) - a_i,\ \frac{p_i a_j}{p_j a_i} - 1,\ \text{etc.}$$

Each of the divisor methods is stable for one of the above measures (Balinski and Young, 1975). Namely, Adam's method is stable for $a_j - a_i\,(p_j/p_i)$,

Dean's for $\frac{p_i}{a_i} - \frac{p_j}{a_j}$, Hill's for $\frac{p_i a_j}{p_j a_i} - 1$,

Webster's for $\frac{a_j}{p_j} - \frac{a_i}{p_i}$ and Jefferson's for $a_j\,(p_i/p_j) - a_i$.

Another approach to select an apportionment method is the optimization of some function. Ideally, one would like to have the a_i "close" to the fair share $s_i = \text{med}\,\{f_i, c_i, \lambda q_i\}$ for every i. So, naturally, one may select that apportionment method that minimizes $\sum_i |a_i - s_i| =$ or $\sum_i (a_i - s_i)^2$. The "error" inherent in a trial apportionment can be measured in other ways. For example, it might be reasonable to minimize one of the following sums:

$$\sum_i \left| \frac{p_i}{a_i} - \frac{p}{h} \right| \text{ or } \sum_i \left(\frac{p_i}{a_i} - \frac{p}{h} \right)^2 \text{ or } \sum_i \left| \frac{a_i}{p_i} - \frac{h}{p} \right| \text{ or } \sum_i \left(\frac{a_i}{p_i} - \frac{h}{p} \right)^2 \text{ or}$$

$$\sum_i p_i \left(\frac{a_i}{p_i} - \frac{h}{p} \right)^2 \text{ or } \sum_i \left(\frac{p_i}{a_i} - \frac{p}{h} \right)^2.$$

Or, we may wish to select that apportionment method that minimizes the "Kullback's information function" $\sum_i a_i * \log \frac{a_i}{s_i}$ or the $\max_i \left| \frac{a_i}{s_i} - 1 \right|$ or the $\max_i \left| \frac{s_i}{Q_i} - 1 \right|$, where the latter is desirable because we do not want to disturb the data. It is clear that there are many more minimum-maximum criteria possible to be used and that each of them may suggest a different apportionment as appropriate and fair.

5. Commeting on the Apportionment Methods Used in Greece and in U.S.A. in Determining Political Representation.

In Greece, from 1926 until today, very rarely one can find two consecutive elections with exactly the same electroral system. In the last several decades, there is a "tradition" of changing the electoral system. Typically the changes are taking place a few months before the elections, so the government's chances for reelection are maximaized. It is worth mentioning that only in 1954 the government decided to vote for a new electoral law in the middle of its term. This is the only electoral law that, although was voted by the parliamentm was never used, since the same governmental majority replaced it by another one a few days before the elections of 1956! This bad tradition was made more official, in a way, when the 1974-1978 government rejected a request of the opposition for a stable electoral system. It may be that, the majority of the Greek politicians view the electoral systems as the most easily manipulated mechanism of the politics. That is why, the electoral system, part of which describes the method of apportionment, is always the center of political discussions, with increasing intensity as the election day approaches. Despite the significance of the electoral system and the fiery discussions about it, there are hardly any scientific monographs, books or papers in Greece that give some analysis of the systems and their characteristics. And, in particular, there is no in-depth study of the methods of apportionment that gevernments have or could have been using. In the Greek bibliography of the last seventy years, the books regarding the elections are limited to the following:

(i) Papanastasiou, A., *Democracy and Electoral System* (1923).
(ii) Georgantas, M., *Regarding proportional representation of the minorities* (1923).
(iii) Haritakis, G, *Regarding proportional election* (1923).
(iv) Meuno, Z., *The political powers of Greece* (1965).
(v) Ralli, K., *Vote, elections and contemporary electoral systems* (1969).
(vi) Pantelis, A., *The Greek electoral systems and the elections from 1926 to* 1985(1988).

Several papers concerning the elections in Greece have been published. These include:

(i) Clogg, R., "Greece" in the Bogdanov, V. and Butler, D., *Democracy and Elections. Electoral Systems and their political consequences,* Cambridge, 1983, 190-208.

(ii) Katsikis, D., *Grece,* in *Guide international des statistiques electorates, Volume 1: Elections nationates en Europe occidentale,* Rokkan, St., and Mayriat, J. (eds), The Hague-Paris, 1969.

(iii) Mourouzis, St., *The proclamation of the deputees during the proceedings of the second apportionment* (in Greek), Law Studies, Vol. I, Komotini, Greece, 1980.

(iv) Nikilakopoulos, I., *The electoral system as a guiding mechanism: Inferences an conclusions {in* Greek), Parliamentary Review, 2nd Issue, October 1989, 22-30.

(v) Pantelis, A., *The electoral system and the small parties,* The constitution, Athens, 1982, 404.

(vi) Papadimitriou, G., *Electoral system and governing system,* Parliamentary Review, 2nd Issue, October 1989, 16-21.

(vii) Vegleris, Ph., *L'evolution dy systeme et des pratiques electorates en Grece,* in Cadart, J., *Les modes du scrutin des dix-huitpays libres de TEurope Occidentale. Leurs resultats et leurs effects compares,* Paris, 1983, 331-353.

All these books and papers are mainly concerned with the effects of the several electoral systems on the operation of the democratic form of government, rather than present a systematic and comparative analysis of the characteristics of the several systems.

Nikolakopoulos, I. (1989) studies the electoral systems that have been applied in some countries, and in particular, analyzes the characteristics of the electoral systems in Greece since 1926. There is no investigation on the methods of apportionment that government have been using to apportion the seats of the parliament to the districts.

Studing the legislative decrees of Greece concerning the election of the deputees and, in particular, the methods of apportionment from 1929 to 1990, we find the following:

It was in 1956 when the Greek legislation first introduced the Hamilton method, in the form that it is stated. The objective was to apportion the seats of the parliament into the electoral districts.

In the years before 1956, the method used was some variation of the Hamilton method. The size of the parliament was most of the times specified (either

250 or 300 seats). There were other times where the size was not specified but rather was being determined by the method used (always though kept around 250 or 300 seats, with the exception of the parliament size of 1946 which was 354 seats). The method used in those years was working as follows:

First, a divisor χ and a remainder r where declared. Each district would receive a number of seats equal to the integer part of the quotient obtained by the division of the population of the district by x. Districts having remainder larger than r would receive an extra seat. Districts having received no seats, would receive one. Whenever the parliament size was specified, then the assigned numbers of seats were being adjusted so that they summed up to the given size. If additional seats were needed then the districts with the largest remainder were given an extra one. If a number of seats needed to be deducted then the districts with the smallest remainders would lose one.

Currently, the Hamilton method is being used to apportion the 288 of the 300 parliamentary seats to the electoral districts (the remaining 12 are given to the parties according to their country-wide percentages received on the election day). Hamilton's approach is simple and direct. It seems reasonable and natural. The major advantage of this method is that it results apportionments that stay within the fair share. Thus minimizing $\Sigma |a_i - s_i|$, $\Sigma(a_i - s_i)^2$ and actually, any ℓ_p-norm of $\vec{a} - \vec{s}$ (Birkhoff, 1976). There are some paradoxes of Hamilton's method. One of them may occur when the size of the parliament changes. Another occurs when districts' populations change.

More analytically, one may notice that although a district may be given x seats when the size of the parliament is y, it is given only x-1 when the size becomes y+1. For example we observe that if 290 seats were to be alloted to the 56 districts (based on the census of 1981) then the district of "Elea" would receive 7 seats while if 291 seats were to be alloted the district of "Elea" would receive only 6 seats (see Table 2).

On the other hand, one may observe that although district A grows faster than district B (and becomes proportionally larger than B), A loses a seat to B. Or even a more striking observation may be that, although a district A which loses people gains a seat, another district which gains people loses a seat. However, this phenomena occurs very rarely.

Another problem with hamilton's approach is that it is not uniform. For example, given the Hamilton-apportionment of the 288 seats to the 56 districts of Greece, say a_i, $i \in I = \{1, ..., 56\}$ (Table 2), we find a subset of districts with populations p_j, $j \in J \subset I$ that when considered alone for apportionment of its corresponding number of seats $\sum_{i \subset J} a_i$, does not receive the preassigned apportionment (Table 3). Specifically, we consider a subset of 10 districts whose original allotment was a total of 93 seats. Applying the Hamilton method to apportion these 93 seats to the 10 districts we observe that the districts of "2nd Athenean communtiy" and "Pieria" receive 33 and 3 seats respectively while their original apportionment was 32 and 4 seats respectively.

Thus, the Hamilton method used in Greece to apportion the seats of the parliament to the districts, although stays within the fair share and its apportionments minimize any ℓ_p- norm of $\vec{a} - \vec{s}$, it is not uniform and it falls into paradoxes that may be very critical. Achieving apportionments that accurately reflect relative changes in populations or changes in the size of the parliament seems more important than always staying within the fair share.

Among the five popular methods, Webster's method is the unique population monotone method that is near fair share interpreted absolutely ($| a_i$-s_i) or relatively $\left(\frac{a_i - s_i}{s_i}\right)$, and it is stable for the test $\frac{a_j}{p_j} - \frac{a_i}{p_i}$.

As one can see in Table 2, the Hamilton's and Webster's apportionments on the basis of the 1981 census, happen to coincide. We also investigate the apportionments given by 1-stationary and 2-stationary methods. The fact that each district should receive at least one seat, makes the divisor point d(0) powerless in both methods. In Table 7, we display the 1-stationary apportionment. Also displayed are the 2-stationary apportionments for selected values of the divisor point $d(1) \subset [0, 1]$.

The 1-stationary apportionment coincides with those of Webster and Hamilton. No district can be brought closer to its fair share without moving another state further from its fair share. On the other hand, the 2-stationary apportionments are slightly different than the 1-stationary one and, consequently, it is possible to take a seat from a state and give it to another and simultaneously bring both of them closer to their fair shares. But, even though 2-stationary fails to stay near fair shares, it rounds in such a way so that the first two moments of the original number and the roundings agree.

In the United States, on the other hand, the method currently being used to apportion the 435 seats of the House of Representatives to the 50 states, is the method of Hill or method of equal proportions. The method of equal proportions minimizes the percentage differences in the proportion (or ratio) of representation in the House among all possible pairs of States, regardless of their size (population). This is true whether representation is calculated on the basis of (a) the number of Representatives per million population, or (b) the population per Representative. This method is being used since 1941 and recently (March 31, 1992), the Supreme Court upheld the constitutionality of the method when Montana asked for an extra seat taken by the State of Washington. As it can be seen in Table 5, Montana would maintain both seats it was given in the decade of 1980 if the apportionment was done with one of the following methods: Dean, Lownders or Adams.

The steps that are being followed in computing the Hill apportionment of the 435 seats of the House of Representatives are:

i) Each state receives one seat.

ii) A priority list is created to determine the assignment to individuals states of each additional seat in the House from the 51st to the 435th. The priority values are calculated as follows:

(1) Multiply the population of each State Successively by the multipliers shown in Table 1 — Multipliers for the Determination of Apportionment Priority Values using as many multipliers as necessary to calculate a priority value for each seat in the House to which the State may be entitled.

(2) Prepare a separate 3x5 card for each priority value calculated, showing the name of the State, the priority value, and the number of Representatives (size of delegation) corresponding to this value.

(3) After completion of the calculation of the priority values for all States, arrange the priority values for all the States (i.e., all the 3x5 cards) in sequence by size from the largest priority value to the smallest.

(4) Number the cards in rank order, beginning with number 51 for the largest priority value through 435. In order words, the 51st Representative is assigned to the State having the largest priority value, the 52nd to the State having the next largest, etc., untill all 435 seats have been allocated.

(5) Prepare an alphabetical listing of the States and working in reverse sequence, beginning with card numbered 435 (corresponding to the assignment of the 435th seat), then card 434, etc., enter opposite each State name the largest number of representatives listed for each. (The first card encountered for each

State will contain the largest "size of delegation" number for that State). Note that entries will be made only for States entitled to two or more Representatives. Enter the number "1" for all remaining States. The sum of the "size of delegation" entries for all States will, of course, add to 435 —the total number of Representatives apportioned.

The Hill-apportionments, on the basis of the census of 1980 and 1990, are shown in Tables 4 and 5, respectively. In the 1990 census, for only the second time since 1900, the Census Bureau allocated the Department of Defense's overseas employees to particular States for reapportionment purposes, using an allocation method that is determined most closely resembling "ususal residence", its standard measure of state affiliation. The 1990 reapportionment, which is based on the populations that include the overseas employees is shown in Table 6. We notice, as Massachusetts did, that by including the overseas employees there was a shift of a seat from the State of Massachusetts to Washington State. Massachusetts appealed to the President and the Secretary of Commerce (April 21,1992) but they had no luck as their appeal was denied (Decision was taken on June 26, 1992).

In Table 6, apart from the Hill apportionment we display the ones due to the Webster, Deam Lowndes and Adams one. As it can be seen in Table 6, Montana would receive two seats if the apportionment was done with one of the following methods: Deam, Lownders or Adams. Note that only Lowndes method justifies Montanta's request for a second seat taken from Washington State.

Hill's method or method of equal proportions does not satisfy fair share, i.e. its apportionments are not always within fair share. Instead it satisfies the principle of pairwise comparisons between states. Namely, Hill's method is stable for the relative difference between the average district sizes $\left(\frac{p_i/a_i - p_j/a_j}{p_j/a_j} = \frac{p_i a_j}{p_j a_i} - 1, \text{ where } \frac{p_i}{a_i} \geq \frac{p_i}{a_i} \right)$. This pairwise comparison may lead to inconsistencies such as a state having fair share an integer number, but receives some other number. Another absurdity of this pairwise comparison occurs in the following example. Assum that 50 representatives are to be apportioned among 50 states, with populations $(10^8, 1, 1, ..., 1)$. Then, the unique Hill-solution is $(1, 1, ..., 1)$, which by coincidence agrees with the Constitutional right of each state to be represented. But this means that 49 people out of a population of hundred million have 98% of the rpresentation, which is unreasonable.

On the other hand, Webster's method, which is stable for the difference $\frac{a_j}{p_j} - \frac{a_i}{p_i}$ would not allow a situation similar to the latter. Also, as Balinski and Young observe, the probability that Webster's method will violate fair share is only about 1 in 1,600 apportionments, where Hill's method is five times as likely. Also, even though Webster's method does not stay within the fair share all of the time, it does stay near the fair share all of the time, whether measured in absolute or relative terms. And, it is the only divisor method that does so.

Comparing the 1990 apportionments of Hill and Webster, based on the population that includes the overseas employees, we find that they differ in two states: Oklahoma and Massachusetts. Webster takes a seat from Oklahoma and gives it to Massachusetts. Observe that Massachusetts would receive an extra seat if either the overseas employees were not included in the population or if, instead of the Hill's method the Webster's method was used for the apportionment.

Note that, if i= Oklahoma and j= Massachusetts, then under Hill's method $\frac{a_i}{p_i} = 0{,}000002102$ and $\frac{a_j}{p_j} = 0.000001658$, i.e., in Oklahoma there are 2.102 representatives for every million people where, in Massachusetts, there are 1.658 representatives for every million people, a difference of 0.443 per million people in favor of Oklahoma. However, under Webster's method, this difference becomes 0.072 per million people in favor of Massachusetts. So, if our goal is to minimize the difference $\frac{a_i}{p_i} - \frac{a_j}{p_j}$, the Webster's method prevails.

On the other hand, as one can see in Table 8, 1-stationary coincides with Webster's apportionment. Probably these two methods would not coincide in the case where a state could receive no seat (and, of course, using an appropriate divisor point d(0) for 1-stationary). The 2-stationary with $0.4 \leq d(1) \leq 0.6$ results exactly the same apportionment with that of Webster's and 1-stationary. With $d(1) \leq 0.3$, 2-stationary helps the very small states, where, with $d(1) \geq 0.7$, it hammers them. Note that, the set of 2-stationary methods contains the set of 1-stationary methods and this, in its turn, contains the Webster method. This fact gives us the opportunity to select the divisor points d(0) and d(1) in [0,1], and, if possible, to succeed in obtaining a better method. This will depend also on the distribution of the population in study, as well as on the criteria based on which one may choose the best method of apportionment.

To summarize, the Hamilton method used in Greece to apportion the seats of the parliament to the districts, although stays within fair share ($\bar{s}$) and its apportionments ($\vec{a}$) minimize any ℓp-norm of $\vec{a} - \vec{s}$, is neither uniform nor house monotone, and it falls into paradoxes that may be of great importance. On the other hand, the Webster's method and 1-stationary method result apportionments that accurately reflect relative changes in population or changes in the size of the parliament. At the same time, very rarely they violate fair share, and they are always near fair share.

In the United States, Hill's method has been the method of apportionment since 1941. It minimizes the difference between the representation in the House of any two states when measured by the relative difference in the average population per district and also by the relative difference in the individual share in a representative. But if the main purpose of apportionment is to give to ny group of individuals as nearly as may be the same weight in choosing representatives in the House whether they happen to live in the large states or the small states, then Webster's method is the one that should prevail. As Balinski and Young observe (1982), Hill's method is five times more likely to violate fair share than Webster's method. The latter, together with the 1-stationary method, are the only divisor methods that stay near the fair share all the time whether measured in absolute or relative terms. Moreover, small shifts in population can lead to large shifts in Hill's apportionment.

The truth is that most of the times Hill's, Webster's and 1-stationary apportionments are identical. But at the times where the only difference is one or two transferred seats, the decision to favor one method over another must be based on logic and on constitutional principles, and not be driven by political interests.

Appendix

TABLE 1
Multipliers for the Determination of Priority Values
(Methods of equal proportions)

n	Multiplier	n	Multiplier	n	Multiplier
2	0.70710678	21	0.04879500	41	0.02469324
3	0.40824829	22	0.04652421	42	0.02409813
		23	0.04445542	43	0.02353104
4	0.28867513				
5	0.22360680	24	0.04256283	44	0.02299002
6	0.18257419	25	0.04082483	45	0.02247333
		26	0.03922323	46	0.02197935
7	0.15430335				
8	0.13363062	27	0.03774257	47	0.02150662
9	0.11785113	28	0.03636965	48	0.02105380
		29	0.03509312	49	0.02061965
10	0.10540926				
11	0.09534626	30	0.03390318	50	0.02020305
12	0.08703883	31	0.03279129	51	0.01980295
		32	0.03175003	52	0.01941839
13	0.08006408				
14	0.07412493	33	0.03077287	53	0.01904848
15	0.06900656	34	0.02985407	54	0.01869240
		35	0.02898855	55	0.01834940
16	0.06454972				
17	0.06063391	36	0.02817181	56	0.01801875
18	0.05716620	37	0.02739983	57	0.01769981
		38	0.02666904	58	0.01739196
19	0.05407381				
20	0.05129892	39	0.02597622	59	0.01709464
		40	0.02531848	60	0.01680732

Note: For the method of equal proportions the multipliers (i.e., the reciprocals of the geometric means of successive numbers) are calculated from the formula $\frac{1}{\sqrt{n(n-1)}}$, where n is the Number of Representatives (size of delegation).

TABLE 2

Comparative Apportionment of Parliament Seats According to Various Methods

Greek Districts	Population	Hamilton's apportionment of 290 seats	Hamilton's apportionment of 291 seats	Hamilton's apportionment of 288 seats	Webster's apportionment of 288 seats
1st Athenean communtiy	715840	21	22	21	21
2nd Athenean copmmunity	83830	32	33	32	32
1st Pirean community	257695	8	8	8	8
2nd Pirean communtiy	273614	8	8	8	8
Etolia & Akarnania	284954	9	9	8	8
Attiki	226527	7	7	7	7
Veotia	127783	4	4	4	4
Evia	211440	6	6	6	6
Evritania	42258	1	1	1	1
Phtiotida	188808	6	6	6	6
Fokida	56674	2	2	2	2
Argolida	98584	3	3	3	3
Arkadia	143782	4	4	4	4
Ahaia	285069	9	9	9	9
Elea	217371	7	6	6	6
Korynthia	135199	4	4	4	4
Lakonia	113042	3	3	3	3
Messinia	218746	7	7	7	7
Zakynthos	37979	1	1	1	1
Kerkyra	110606	3	3	3	3
Kephalenia	46165	1	1	1	1
Leukada	31088	1	1	1	1
Arta	106492	3	3	3	3
Thesprotia	54364	2	2	2	2
Yannina	187460	6	6	6	6
Preveza	71319	2	2	2	2
Karditsa	179148	5	5	5	5
Larissa	263134	8	8	8	8
Magnisia	186771	6	6	6	6
Trikala	171761	5	5	5	5
Grevena	52658	2	2	2	2
Drama	119115	4	4	4	4
Hemathia	139209	4	4	4	4
1st Section of Saloniki	420833	13	13	13	13
2nd Section of Saloniki	218156	7	7	7	7
Kavala	150389	4	4	4	4
Kastoria	52076	2	2	2	2
Kilkis	107786	3	3	3	3
Kozani	167382	5	5	5	5
Pella	154990	5	5	5	5
Pieria	118354	4	4	4	4
Serres	264777	8	8	8	8
Florina	63029	2	2	2	2
Halkidiki	97777	3	3	3	3
Evros	161300	5	5	5	5
Xanthi	97990	3	3	3	3
Rodopi	114545	3	3	3	3
Dodekanisa	134654	4	4	4	4
Cyclades	115369	3	3	3	3
Lesvos	128472	4	4	4	4
Samos	49380	1	1	1	1
Khios	60315	2	2	2	2
Iraklio	249302	7	7	7	7
Lasythi	82222	2	2	2	2
Rethymno	81051	2	2	2	2
Khania	137504	4	4	4	4
TOTAL	9666138	290	291	288	288

TABLE 3
Comparative Apportionment of Parliament Seats According to Various Methods

District	Population based on 1981 census	Hamilton's apportionment of 93 seats	Corresponding numbers from Hamilton's apportionment of 288 seats of 56 districts
1st Athenean community	715840	21	21
2nd Athenean communtiy	1083830	33	32
1st Pirean community	257695	8	8
2nd Pirean community	273614	8	8
Kastoria	52076	2	2
Kilkis	107786	3	3
Pella	154990	5	5
Pieria	118354	3	4
Serres	264777	8	8
Florina	63029	2	2
TOTAL	3091991	93	93

TABLE 4
Comparative Apportionment of Parliament Seats According to Various Methods

States of USA	Population in 1980	Hill's apportionment in 1980	Webster's apportionment in 1980
Alabana	3890061	7	7
Alaska	400481	1	1
Arizona	2717866	5	5
Arkansas	2285513	4	4
California	23668562	45	45
Colorado	2888834	6	6
Connecticut	3107576	6	6
Delaware	595225	1	1
Florida	9739992	19	19
Georgia	5464265	10	10
Hawaii	965000	2	2
Idaho	943935	2	2
Illinois	11418461	22	22
Indiana	5490179	10	11
Iowa	2913387	6	6
Kansas	2363208	5	5
Kentucky	3661433	7	7
Louisiana	4203972	8	8
Maine	1124660	2	2
Maryland	4216446	8	8
Massachusetts	5737037	11	11
Michigan	9258344	18	18
Minnesota	4077148	8	8
Mississippi	2520638	5	5
Missouri	4917444	9	9
Montana	786690	2	2
Nebraska	1570006	3	3
Nevada	799184	2	2
New Hampshire	920610	2	2
New Jersey	7364158	14	14
New Mexico	1299968	3	2
New York	17557288	34	34
North Carolina	5874429	11	11
North Dakota	652695	1	1
Ohio	10797419	21	21
Oklahoma	3025266	6	6
Oregon	2632663	5	5
Pennsylvania	11866728	23	23
Rhode Island	947154	2	2
South Carolina	3119208	6	6
South Dakota	690178	1	1
Tennessee	4590750	9	9
Texas	14228383	27	27
Utah	1461037	3	3
Vermont	511456	1	1
Virginia	5346279	10	10
Washington	4130163	8	8
West Virginia	1949644	4	4
Wisconsin	4705335	9	9
Wyoming	470816	1	1
TOTAL	3890061	435	435

TABLE 5
Comparative Apportionment of Parliament Seats According to Various Methods

States of USA	Population in 1990	Hill's apportionment in 1990	Webster's apportionment in 1990	Dean's apportionment	Lownde's apportionment	Adam's apportionment
Alabana	4040587	7	7	7	7	7
Alaska	550043	1	1	1	1	1
Arizona	3665228	6	6	7	6	7
Arkansas	2350725	4	4	4	4	4
California	29760021	52	52	50	52	50
Colorando	3294394	6	6	6	6	6
Connecticut	3287116	6	6	6	6	6
Delaware	666168	1	1	2	2	2
Florida	12937926	23	23	22	22	22
Georgia	6478216	11	11	11	11	11
Hawaii	1108229	2	2	2	2	2
Idaho	1006749	2	2	2	2	2
Illinois	11430602	20	20	19	20	19
Indiana	5544159	10	10	10	10	10
Iowa	2776755	5	5	5	5	5
Kansas	2477572	4	4	5	5	5
Kentucky	3685296	6	6	7	7	7
Louisiana	4219973	7	7	8	7	8
Maine	1227928	2	2	3	3	3
Maryland	4781468	8	8	8	8	8
Massachusetts	6016425	11	11	10	10	10
Michigan	9295297	16	16	16	16	16
Minnesota	4375099	8	8	8	8	8
Mississippi	2573216	5	4	5	5	5
Missouri	5117073	9	9	9	9	9
Montana	799065	1	1	2	2	2
Nebraska	1578385	3	3	3	3	3
Nevada	1201833	2	2	2	2	2
New Hampshire	1109252	2	2	2	2	2
New Jersey	7730188	13	14	13	13	13
New Mexico	1515069	3	3	3	3	3
New York	17990455	31	31	30	31	30
North Carolina	6628637	12	12	11	11	11
North Dakota	638800	1	1	2	2	2
Ohio	10847115	19	19	18	19	18
Oklahoma	3145585	6	5	6	6	6
Oregon	2842321	5	5	5	5	5
Pennsylvania	11881643	21	21	20	20	20
Rhode Island	1003464	2	2	2	2	2
South Carolina	3486703	6	6	6	6	6
South Dakota	696004	1	1	2	2	2
Tennessee	4877185	9	9	9	8	9
Texas	16986510	30	30	29	29	29
Utah	1722850	3	3	3	3	3
Vermont	562578	1	1	1	1	1
Virginia	6187358	11	11	11	11	11
Washington	4866692	8	9	9	8	9
West Virginia	1793477	3	3	3	3	3
Wisconsin	4891769	9	9	9	8	9
Wyoming	453588	1	1	1	1	1
TOTAL	248102973	435	435	435	435	435

TABLE 6
Comparative Apportionment of Parliament Seats According to Various Methods

States of USA	Population in 1990 includ-ing overseas	Hill's apportionment in 1990	Webster's apportionment in 1990	Dean's apportionment	Lownde's apportionment	Adam's apportionment
Alabana	4062608	7	7	7	7	7
Alaska	551947	1	1	1	1	1
Arizona	3677985	6	6	7	6	7
Arkansas	2362239	4	4	4	4	4
California	29839250	52	52	50	52	50
Colorando	3307912	6	6	6	6	6
Connecticut	3295669	6	6	6	6	6
Delaware	668696	1	1	2	2	2
Florida	13003362	23	23	22	22	22
Georgia	6508419	11	11	11	11	11
Hawaii	1115274	2	2	2	2	2
Idaho	1011986	2	2	2	2	2
Illinois	11466682	20	20	19	20	19
Indiana	5564228	10	10	10	10	10
Iowa	2787424	5	5	5	5	5
Kansas	2485600	4	4	5	5	5
Kentucky	3698969	6	6	7	7	7
Louisiana	4238216	7	7	8	7	8
Maine	1233223	2	2	3	3	3
Maryland	4798622	8	8	8	8	8
Massachusetts	6029051	10	11	10	10	10
Michigan	9328784	16	16	16	16	16
Minnesota	4387029	8	8	8	8	8
Mississippi	2586443	5	5	5	5	5
Missouri	5137804	9	9	9	9	9
Montana	803655	1	1	2	2	2
Nebraska	1584617	3	3	3	3	3
Nevada	1206152	2	2	2	2	2
New Hampshire	1113915	2	2	2	2	2
New Jersey	7748634	13	14	13	13	13
New Mexico	1521779	3	3	3	3	3
New York	18044505	31	31	30	31	30
North Carolina	6657630	12	12	11	11	11
North Dakota	641364	1	1	2	2	2
Ohio	10887325	19	19	18	19	18
Oklahoma	3157604	6	5	6	6	6
Oregon	2853733	5	5	5	5	5
Pennsylvania	11924710	21	21	20	20	20
Rhode Island	1005984	2	2	2	2	2
South Carolina	3505707	6	6	6	6	6
South Dakota	699999	1	1	2	2	2
Tennessee	4896641	9	9	9	8	9
Texas	17059805	30	30	29	29	29
Utah	1727784	3	3	3	3	3
Vermont	564964	1	1	1	1	1
Virginia	6216568	11	11	11	11	11
Washington	4887941	9	9	9	8	9
West Virginia	1801625	3	3	3	3	3
Wisconsin	4906745	9	9	9	8	9
Wyoming	455975	1	1	1	1	1
TOTAL	249022783	435	435	435	435	435

TABLE 7
Comparative Apportionment of Parliament Seats According to Various Methods

				2-stationary		
Greek Districts	**Fair Share**	**1-stationary**	**d(1)=.2**	**d(1)=.4**	**d(1)=.6**	**d(1)=.8**
1st Athenean communtiy	21.322783	21	21	21	21	21
2nd Athenean copmmunity	32.28413	32	32	32	32	33
1st Pirean community	7.675981	8	8	8	8	8
2nd Pirean communtiy	8.150162	8	8	8	8	8
Etolia & Akarnania	8.487947	8	8	8	9	9
Attiki	6.747578	7	7	7	7	7
Veotia	3.806282	4	4	4	4	4
Evia	6.298818	6	6	6	6	6
Evritania	1.258742	1	2	1	1	1
Phtiotida	5.624039	6	6	6	6	6
Fokida	1.688153	2	2	2	2	1
Argolida	2.936529	3	3	3	3	3
Arkadia	4.282846	4	4	4	4	4
Ahaia	8.491373	9	8	8	9	9
Elea	6.474847	6	6	6	7	7
Korynthia	4.027183	4	4	4	4	4
Lakonia	3.367191	3	3	3	3	3
Messinia	6.515804	7	6	7	7	7
Zakynthos	1.131283	1	1	1	1	1
Kerkyra	3.29463	3	3	3	3	3
Kephalenia	1.375121	1	2	1	1	1
Leukada	1	1	1	1	1	1
Arta	3.172086	3	3	3	3	3
Thesprotia	1.619345	2	2	2	2	1
Yannina	5.583886	6	6	6	6	6
Preveza	2.124385	2	2	2	2	2
Karditsa	5.336296	5	5	5	5	5
Larissa	7.837993	8	8	8	8	8
Magnisia	5.563363	6	6	6	6	6
Trikala	5.116259	5	5	5	5	5
Grevena	1.568528	2	2	2	1	1
Drama	3.548088	4	4	4	4	4
Hemathia	1.14663	4	4	4	4	4
1st Section of Saloniki	12.535386	13	13	13	13	13
2nd Section of Saloniki	6.49823	7	6	7	7	7
Kavala	4.479649	4	4	4	4	5
Kastoria	1.551192	2	2	2	1	1
Kilkis	3.21063	3	3	3	3	3
Kozani	4.985821	5	5	5	5	5
Pella	4.616699	5	5	5	5	5
Pieria	3.52542	4	4	4	4	4
Serres	7.886934	8	8	8	8	8
Florina	1.87745	2	2	2	2	2
Halkidiki	2.912491	3	3	3	3	3
Evros	4.804656	5	5	5	5	5
Xanthi	2.918836	3	3	3	3	3
Rodopi	3.411961	3	3	3	3	3
Dodekanisa	4.010494	4	4	4	4	4
Cyclades	3.436506	3	3	3	3	3
Lesvos	3.826806	4	4	4	4	4
Samos	1.470886	1	2	2	1	1
Khios	1.796608	2	2	2	2	2
Iraklio	7.425978	7	7	7	7	7
Lasythi	2.449153	2	2	2	2	2
Rethymno	2.414273	2	2	2	2	2
Khania	4.095843	4	4	4	4	4
TOTAL		288	288	288	288	

TABLE 8
Comparative Apportionment of Parliament Seats According to Various Methods

States of USA	Fair Share	1-stationary	d(1)=.2	2-stationary d(1) = .4	d(1) = .6	d(1) = .8
Alabana	4.092533	7	7	7	7	7
Alaska	1	1	1	1	1	1
Arizona	6.421055	6	6	6	6	6
Arkansas	4.124016	4	4	4	4	4
California	52.0936	52	52	50	52	50
Colorando	5.774979	6	6	6	6	6
Connecticut	5.753605	6	6	6	6	6
Delaware	1.167415	1	1	1	1	1
Florida	22.701373	23	23	23	23	23
Georgia	11.36245	11	11	11	11	11
Hawaii	1.947054	2	2	2	2	2
Idaho	1.766733	2	2	2	2	1
Illinois	20.018625	20	20	20	20	20
Indiana	9.714074	10	10	10	10	10
Iowa	4.866307	5	5	5	5	5
Kansas	4.339938	4	4	4	4	4
Kentucky	6.45769	6	6	6	6	6
Louisiana	7.399111	7	7	7	7	7
Maine	2.152971	2	2	2	2	2
Maryland	8.377472	8	8	8	8	8
Massachusetts	10.525565	11	10	11	11	11
Michigan	16.286265	16	16	16	16	16
Minnesota	7.65891	8	8	8	8	8
Mississippi	4.515433	5	4	5	5	5
Missouri	8.969619	9	9	9	9	9
Montana	1.403027	1	2	1	1	1
Nebraska	2.766437	3	3	3	3	3
Nevada	2.10571	2	2	2	2	2
New Hampshire	1.944682	2	2	2	2	2
New Jersey	13.527627	13	13	13	13	14
New Mexico	2.656734	3	3	3	3	3
New York	31.50224	31	31	30	31	30
North Carolina	11.622943	12	12	12	12	12
North Dakota	1.119698	1	1	1	1	1
Ohio	19.007179	19	19	19	19	19
Oklahoma	5.51257	5	5	5	5	6
Oregon	4.982207	5	5	5	5	5
Pennsylvania	20.818254	21	21	21	21	21
Rhode Island	1.756255	2	2	2	2	1
South Carolina	6.120291	6	6	6	6	6
South Dakota	1.222064	1	2	1	1	1
Tennessee	8.548595	9	9	9	9	9
Texas	29.783143	30	30	30	30	30
Utah	3.016379	3	3	3	3	3
Vermont	1	1	1	1	1	1
Virginia	10.852934	11	11	11	11	11
Washington	8.533406	9	9	9	9	9
West Virginia	3.145291	3	3	3	3	3
Wisconsin	8.566234	9	9	9	9	9
Wyoming	1	1	1	1	1	1
TOTAL		435	435	435	435	435

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