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THE COST FUNCTION FOR THE PREVENTIVE-MAINTENANCE REPLACEMENT PROBLEM

By

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Abstract

Let a discounted continuous review preventive-maintenance replacement model be such that its total discounted cost is given by means of two functional equations. We assume that downtime is caused by equipment breakdowns, and the length of a given downtime is the time necessary to repair the equipment and set it back in operation. The periodic preventive replacement policy is to replace the equipment by a new identical equipment when service age X is reached, or when the equipment fails. (JEL M11)

1. Introduction

Let us consider an system made up of only one component of age x, x>0. At the time of the study the equipment is operating correctly. Its breakdown follows a probabilistic law and it is controlled by continuous inspection. We wish to obtain the optimal periodic policy of preventive replacement for the chosen equipment.

Through time if the equipment does not have a breakdown, we face two alternatives:

i) do nothing, that is, keep the equipment in service.
ii) perform a preventive replacement.

If the equipment fails, it is replaced immediately.

In both cases the equipment is inactive during the replacement time, incurring a cost due to the loss of operating time.

In a preventive replacement model each replacement serves as a time origin for the process. So, the optimal policy of replacement will be periodic and it will be distinguished by a single number, X^*, which will remain constant throughout the process. X^* represents the optimum replacement for the equipment.

Experience and common sense demonstrate that the economic life can never surpass the service age. Mathematically, this result can be obtained when the annual rate of increase is very high, an event which occurs in inflationary situations.

The optimal policy required should reflect this periodic characteristics. Then, we have that for an equipment of initial age x and age per time units x_t, this policy includes the following rules:

- if the equipment fails, replace it inmediately,
- the equipment does not fail,
  - if $x_t < X^*$, keep it in service,
  - if $x_t \geq X^*$, replace it.

In the statement and the following development of the problem, the horizon of the process of investment is taken as infinite. All costs incurred are treated as constant monetary units. In other words, we do not consider the effect inflation has upon them.

2. Variables

Let C_0 represent the sum of all the cost involved in the acquisition, transportation and installation of the new equipment, and let $C(x_t)$ be the operating cost for an equipment aged x_t. $C(x_t)$ is a continuous and strictly increasing function.

During the time used to replace the equipment it will be out of service. Therefore, we will incur in a different cost than the operations cost.

Let

C_0	:	fixed cost when replacing the equipment,
$C(x_t)$	:	operating cost for the equipment aged x_t,
$P(x_t$	:	cost associated with one unit of downtime for an equipment aged x_t,
τ_0 (t)	:	pfd of the necessary time for replacement when the equipment is in an operating condition,
$R_o(x_t)$	:	salvage value for an equipment aged x_t, when the equipment is in an operating condition,

τ_{00} (t) : pfd of the necessary time for replacement when the equipment has failed,

R_{00} (x_t) : salvage value for the equipment aged x_t, when the equipment has failed,

$\lambda(x)$: failure rate function, and

i : interest rate measured over continuous time.

We also understand that $R_0(x_t)$ and $R_{00}(x_t)$ are non-increasing funcrions, because the salvage value of the equipment does not increase with age. $P(x_t)$ is a non-decreasing function, since the cost of losing a time unit of operation does not decrease with the age of the equipment. We assume that the all former functions are continuous and $P(x_t)$, $R_0(x_t)$ and $R_{00}(x_t)$ are differentiables.

Let $\lambda(x_t)\Delta(x_t)$ be the failure rate of the equipment during the interval $(x_t, x_t + \Delta x_t)$, that is, the probability of having a breakdown in that interval, given the equipment has reached the age x_t. We suppose that $\lambda(x_t)$ is a continuous and increasing function of x_t, and we denote by i the interest rate on all the costs incurred.

3. Objective Function

The objective function to minimize to find the value for X* is the total cost of the equipment during its infinite life discounted at the rate i.

i) The Equipment is Kept in Service

Let $g_0(x)$ be the total discounted cost of an equipment, that has an age x, $x \geq 0$. The equipment is functioning correctly and it will not be replaced by new equipment until it breaks down.

The value of $g_0(x)$ will be the operating cost of the equipment for an age x, plus the cost of continuing to use the equipment if it does not fail in the interval $(x, x + \Delta x)$, and also the cost of replacing it, if it fails in the same interval.

If $f_0(x)$ represents the total discounted cost of an equipment that breaks down at age x, the function $g_0(x)$ will be expressed as the functional equation:

$$g_0(x) = C(x)\,\Delta x + (1 - i\Delta t)\,\{1 - \lambda(x)\Delta x)\,g_0\,(x + \Delta x)\} +$$

$$+ (1 - i\Delta t)\,\lambda(x)\,\Delta x\,f_0\,(x + \Delta x) \qquad (1)$$

We assume that the equipment ages continuously in time, aging is measured in time units and is independent of the age of the equipment and time origin we select, so that:

$$\frac{\Delta t}{\Delta x} = 1$$

Then, we have for $g_0(x)$ the relation:

$$g_0(x) = C(x)\Delta x + (1 - i\Delta x)\{1 - \lambda(x)\, \Delta x\, g_0\, (x + \Delta x)\} + \\ + (1 - i\Delta x)\, \lambda(x)\, \Delta x\, f_0\, (x + \Delta x) \qquad (2)$$

Since $C(x)$ and $\lambda(x)$ are valued-positive and continuous functions; therefore, *to determine the behaviour of the function* $g_0(x)$ *it is necessary to study the* properties of the function $f_0(x)$. From the definition of $f_0(x)$ we observe, that it will behave in the same manner as the function of the cost in the case in which we decide to make a replacement due to the age of the equipment.

ii) The Existing Equipment is Replaced by a New One

If a replacement occurs at the time of the study, the total discounted cost will be the sum of the acquisition cost plus the cost of setting up the new equipment minus the salvage value.

Let $g_1(x)$ be the total discounted cost of the equipment in this case. Its value is given by the functional equation:

$$g_1(x) = C_0 - R_0(x) + \\ + \int_0^{\infty} \left[\int_0^{\infty} P(x)\, \exp\, (-it)\, dt + g_0\, (0)\, \exp\, (-iv) \right] \tau_0(v)\, dv \qquad (3)$$

The salvage value of the equipment, $R_0(x)$, is a non-increasing function of the age; therefore, $g_1(x)$ will be also a non-decreasing function. Moreover, the salvage value of the equipment will never exceed the acquisition cost, so that:

$$g_1\,(x) \geq 0$$

Theorem 3.1. *The functions $g_0(x)$ and $g_1(x)$ defined $\forall x \in R^+$ have always positive values and are continuous for every value of x such that $x \in R^+$.*

PROOF:

The function $g_1(x)$ is always positive. From the relation between this function and $f_0(x)$ we have that $f_0(x)$ will also be positive. Therefore, $g_0(x)$ will behave in the same way.

Considering the condition of continuity imposed on all the functions involved in the model, we infer that both function $g_0(x)$ and $g_1(x)$ will be continuous.

Theorem 3.2. *Given an equipment of initial age x, $x \geq 0$, that is operating correctly, its optimal periodic replacement age, X^*, will be the smallest positive root of the equation:*

$$g_0(x) = g_1(x)$$

PROOF:

If the equipment is new, that is, its initial age is $x = 0$, the value $g_0(0)$ will be the total cost of beginning to operate at that time with the equipment. The function $g_1(0)$ will include besides the cost of the acquisition and installation. Then, we have that:

$$g_0(0) < g_1(0) \tag{4}$$

The optimal policy of replacement will depend on the values of this two functions, $g_0(x)$ and $g_1(x)$. Then:

— if the equipment is kept in service: $g_0(x) < g_1(x)$.
— if the equipment is kept in service: $g_0(x) > g_1(x)$.

As the function $g_1(x)$ is a non-decreasing function, the optimum replacement age of the equipment, X^*, will be the smallest positive root of the equation:

$$g_0(x) = g_1(x).$$

Theorem 3.3. *Let $f(x, X^*)$ be the minimum total discounted cost of the equipment when an optimal policy of replacement, defined by X^*, is followed. If the initial age of the equipment satisfies $0 \leq x \leq X^*$, then $f(x, X^*)$ is given by the functional equations:*

$$f(x, X^*) = \begin{cases} C(x)\Delta x + (1 - i\Delta x)\, \lambda(x)\Delta x\, f_0\,(x + \Delta x) + \\ \quad + (1 - i\Delta x)\,(1 - \lambda(x)\,\Delta x\, f(x + \Delta x, X^*) & \text{if } 0 \leq x < X^*, \\ C_0 - R_0(x) + \int_0^{\infty} \Big[\int_0^{v} P(x) \exp(-it)\, dt + \\ \quad + f(0, X^*) \exp(-iv) \Big] \tau_0(v)\, dv & \text{if } x = X^*, \end{cases} \tag{5}$$

PROOF:

Let X^* be the optimum periodic preventive replacement age, then the minimum discounted cost is:

$$f(x, X^*) = \min g_0(x), g_1(x) \tag{6}$$

Taking into account the condition imposed on the initial age of the equipment, $0 \leq x \leq X^*$, and the relation between $g_0(x)$ and $g_1(x)$, we have that:

$$f(x, X^*) = \begin{cases} g_0(x) & \text{if } 0 \leq x < X^* \\ g_1(x) & \text{if } x = X^* \end{cases} \tag{7}$$

therefore, for an equipment of initial age x, $0 \leq x < X^*$, that is functioning correctly, $f(x, X^*)$ is given by (5).

4. The Total Cost Function

Let $X = X^*$ the replacement age and $\varphi_1(x)$, the function of one variable —the initial age of the equipment— that results from fixing the value of X as X^* in the function $f(x, X)$, namely

$$\varphi_1(x) = f(x, X^*)$$

Then, $\varphi_1(x)$ will represent the total discounted cost of an equipment of initial age x and optimal periodic preventive replacement age $X = X^*$.

Theorem 4.1. *The function $\varphi_1(x)$ defined $\forall x \in [0, X^*]$ by the equation:*

$$\varphi_1(x) = \begin{cases} C(x)\Delta x + (1 - i\Delta x)\, \lambda(x)\Delta x\, f_0(x + \Delta x) + \\ \quad + (1 - i\Delta x)\,(1 - \lambda(x)\, \Delta x\, f(x + \Delta x, X^*) & \text{if } 0 \leq x < X^*, \\ C_0 - R_0(x) + \int_0^{\infty} \left[\int_0^{v} P(x) \exp(-it)\, dt + \right. \\ \quad \left. + f(0, X^*) \exp(-iv) \right] \tau_0(v)\, dv & \text{if } x = X^*, \end{cases} \tag{1}$$

is continuous in its whole domain.

PROOF:

From the Theorem 3.1 we know that the functions $g_0(x)$ and $g_1(x)$ are continuous $\forall x \geq 0$, and in particular at the points of intersection of both functions. Therefore, $g_0(x)$ and $g_1(x)$ will also be continuous at $x = X^*$. Then, because

$f(x, X^*) = \varphi_1(x)$ is made up of at least two continuous functions, it will also be continuous in its whole domain.

Theorem 4.2. *The optimal periodic preventive replacement age of an equipment is independent of its age and time.*
PROOF:

If the initial age of the equipment is $0 \leq x < X^*$, the total discounted cost of the equipment is given by:

$$\varphi_1(x) = C(x)\,\Delta x + (1 - i\Delta x)\,\lambda(x)\,\Delta x f_0(x + \Delta x) +$$

$$+ (1 - i\Delta x)\,(1 - \lambda(x)\,\Delta x)\,\varphi_1(x + \Delta x)$$

Hence, rearranging the equation, dividing by Δx and taking limits as $\Delta x \to 0$ we obtain:

$$0 = C(x) + \lambda(x) f_0(x) + \frac{d\varphi_1(x)}{dx} - i + \lambda(x)\,\varphi_1(x) \qquad (2)$$

Defining the function $j(x) = i + \lambda(x)$, substituting it in (2) and solving for the derivate of $\varphi_1(x)$ have:

$$\frac{d\varphi_1(x)}{dx} = j(x)\varphi_1(x) - [\, C(x) + \lambda(x) f_0(x) \,] \qquad (3)$$

If the equipment fails in the interval $(x, x + \Delta x)$, the total discounted cost after that instant is given by the functional equation:

$$f_0(x) = C_0 - R_{00}(x) + \int_0^{\infty} \Big[\int_0^{v} P(x) \exp(-it)\, dt +$$

$$+ \varphi_1(0) \exp(-iv) \Big]\, \tau_{00}(v)\, dv \qquad (4)$$

Since that $\tau_{00}(v)$ is a probability density function and $T_{00}(i)$ is its Laplace transform, integrating it respect to t, results

$$f_0(x) = C_0 - R_{00}(x) + \frac{P(x)}{i} + \Big[-\frac{P(x)}{i} + \varphi_1(0)\, T_{00}(i) \Big] \qquad (5)$$

Let

$$K_{00}(x) = C_0 - R_{00}(x) + \frac{P(x)}{i} [\, 1 - T_{00}(i) \,] \qquad (6)$$

Substituting in (5) we have:

$$f_0(x) = K_{00}(x) + \varphi_1(0)\, T_{00}(i) \qquad (7)$$

Substituting $f_0(x)$ en (3) we obtain:

$$\frac{d\varphi_1(x)}{dx} = j(x)\varphi_1(x) - \{ C(x) + \lambda(x) [K_{00}(x) + \varphi_1(0)\, T_{00}(i)] \} \tag{8}$$

This expression is a linear differential equation. To solve it, we do the following change of variable:

$$\varphi_1(x) = z(x) \exp [J (x)] \tag{9}$$

where $J(x) = \int_0^x j(s)\, ds$, with the initial condition $z(0) = \varphi_1(0)$.

Differentiating (9) with respect to x, matching the obtained expression with (8), simplifying and integrating, we obtain:

$$z(x) - z(0) = \int_0^{\infty} \exp [-J(s)] \Big\{ - \Big[C(s) + \lambda(s) [K_{00}(s) + \varphi_1(0)T_{00}(i)] \Big] \Big\} ds \tag{10}$$

Undoing the change of variable, solving for $\varphi_1(x)$, and expanding the terms of the integral, we have that:

$$\begin{aligned} \varphi_1(x) = {} & \varphi_0(0) \exp [J(x)] - \\ & - \exp [J(x)] \int_0^x \exp [-J(s)] \Big[C(s) + \lambda(s)\, K_{00}(s) \Big] ds - \\ & - \exp [J(x)]\, \varphi_1(0)T_{00}(i) \int_0^x \exp [-J(s)]\, \lambda(s)\, ds \end{aligned} \tag{11}$$

We define the auxiliary functions:

$$\hat{C}(s) = \frac{C(s) + \lambda(s)\, K_{00}(s)}{j(s)} \quad \text{and} \quad \bar{\lambda}(s) = \frac{\lambda(s)}{j(s)} \tag{12}$$

Then, the expression (11) will be written as:

$$\begin{aligned} \varphi_1(x) = {} & \varphi_0(0) \exp [J(x)] - \\ & - \exp [J(x)] \int_0^x \hat{C}(s)\, d\{ - \exp [-J(s)] \} - \\ & - \exp [J(x)]\, \varphi_1(0)\, T_{00}(i) \int_0^x \bar{\lambda}(s)\, d\{ - \exp [-J(s)] \} \end{aligned} \tag{13}$$

Hence, it is evident that independently of the initial age of equipment, its total discounted cost depends on its replacement age, fixed only on basis of the cost of the new equipment.

Corollary 4.1. *The functions $f(x, X)$ and $f(0, X)$ will reach a minimum for same value of $X = X^*$.*

PROOF:

This follow immediately from the Theorem 4.2.

Theorem 4.3. *The total discounted cost of an equipment given by the functional equations (3.5) can be expressed in an equivalent form by:*

$$\varphi_1(0) = \frac{K_0(X^*) \exp[-J(X^*)] + \int_0^{X^*} \hat{C}(s)\, d\{-\exp[-J(s)]\}}{1 - T_o(i) \exp[-J(X^*)] - T_{00}(i) \int_0^{X^*} \lambda(s)\, d\{-\exp[-J(s)]\}}$$

PROOF:

Developing independently each of the functional equations (3.5), that are expressed in terms of the relations between x and X, we have:

i) If $0 \leq x < X^*$, the total discointed cost will be given by the functional equation (4.13).

ii) If $x = X^*$ we make a preventive replacement and $\varphi_1(X^*)$ is given by (4.1).

Since that $\tau_0(v)$ is a probability density function and $T_o(i)$ is its Laplace transform, integrating with respect to t, the above expression will be written as:

$$\varphi_1(X^*) = C_0 - R_0(X^*) + \frac{P(X^*)}{i} + \left[-\frac{P(X^*)}{i} + \varphi_1(0) \right] T_0(i) \qquad (14)$$

Like before, we let:

$$K_0(X^*) = C_0 - R_0(X^*) + \frac{P(X^*)}{i} + [1 - T_0(i)] \qquad (15)$$

that substituted in (14) yields:

$$\varphi_1(X^*) = K_0(X^*) + \varphi_1(0)\, T_0(i) \qquad (16)$$

Because of the continuity of the function $\varphi_1(x)$ in $[0, X^*]$, the expressions (13) and (16) are equal for $x = X^*$, that is:

$$\begin{aligned} K_0(X^*) + \varphi_1(0)\, T_0(i) = \\ = \varphi_1(0) \exp[J(X^*)] - \exp[J(X^*)] \int_0^{X^*} \hat{C}(s)\, d\{-\exp[-J(s)]\} - \\ - \exp[J(X^*)]\, \varphi_1(0)\, T_{00}(i) \int_0^{X^*} \lambda(s)\, d\{-\exp[-J(s)]\} \end{aligned} \qquad (17)$$

Grouping terms and solving for $\varphi_1(0)$, we have:

$$\varphi_1(0) = \frac{K_0(X^*) \exp[-J(X^*)] + \int_0^{X^*} \hat{C}(s)\, d\{-\exp[-J(s)]\}}{1 - T_0(i) \exp[-J(X^*)] - T_{00}(i) \int_0^{X^*} \lambda(s)\, d\{-\exp[-J(s)]\}} \tag{18}$$

References

Baazara, M. and Sheety, JR. (1982). Non-Linear Programming. New York: John Wiley and Sons.

Barlow, R. E. and Hunter, L. C. (1960). Optimum Preventive Maintenance Policies", Oper. Res., 8, 90-100.

Barlow, R.E. and Proschan, F. (1962). "Planned Replacement", In Studies in Applied Probability and Management Science, [Arrow, K. J., Karlin, S., and Scarf, H. (editors)], Stanford University Press, Stanford, Cal.

Barlow, R.E. and Proschan, F. (1965). Mathematical Theory of Reliability. New York: John Wiley and Sons.

Barlow, R.E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing. New York: Holt, Rinehart and Winston.

Elguea, M.T. (1982). "Modelos Aleatorios de Reemplazamiento". Gaceta Mat., 39, 9-20.

Elguea, M.T. (1984). "Modelos Aleatorios de Reemplazamiento de Revision Continua". Estadist. Espanola,26, 105, 129-142.

Elguea, M.T. and Vilaplana, J. P. (1982). "Modelos Aleatorios de Reemplazamiento de Revision Periodica", Acta Salmanticensia - Ciencias, 46, 605-609.

Jorgenson, D.W., McMall, J.J. andRadner, R.. (1967). Optimal Replacement Policy. Amsterdam: North-Holland Publishing Co.

Nakagawa, T. (1980). "A Summary of Imperfect Preventive Maintenance Policies with Minimal Repair", RAIRO Rech. Oper., 14,249-255,(1980).

Nakagawa, T. (1981). "A Summary of Periodic Replacement with Minimal Repair at Failure", J. Oper. Res. Soc. Japan, 24, 213-227.

Nakagawa, T. (1986). "Periodic and Sequential Preventive Maintenance Policies". J. Appl. Probab., 23, 536-542.

Sivazlian, B.D. (1973). "On a Discounted Replacement Problem with Arbitrary Repair Time Distribution", Management Sci., 19, 11, 1301-1309.

Vilaplana, J.P. (1983). Modelos de Reemplazamiento. Rosario, Argentina: Editorial Universitaria, (1983).

Vilaplana, J.P. (1987). "Sobre el Reemplazamiento de Equipos", In Investigacion Operativa IO'87" [J. P. Vilaplana (Editor), Universidad del Pais Vasco, Bilbao], 261-290.

Vilaplana, J.P. (1990). "A Note on Strict Pseudo-Convexity". (Not published).

Vilaplana, J. P. (1993). "Reemplazamiento de Equipos e Inflation". (To appear in Investogaci-óη Operativa - Brasil).

Vilaplana, J. P. and Elguea, M. T. (1990). "Un Modelo de Reemplazamiento Preventivo de Revision Continua con Politica Periodica de Reemplazamiento", Rev. Real Acad. Cien. Exact. Fis. Natur., Madrid, 84, 1, 35-52.

Weiss, G. (1956). "On the Theory of Replacement of Machinery with a Random Failure Tima", Naval Res. Logist., Quart., 3, 4, 279-293.