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A NOTE ON THE UNIT ROOT TEST BASED ON THE SAMPLE AUTOCORRELATIONS

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Abstract

This paper examines the behavior of the proposed by Bierens (1993) unit root test based on the sample autocorrelations when the true generating process contains one moving average term. The performance of the test is investigated first by applying the test to the means of the sample autocorrelations obtained as a ratio of two quadratic forms in normal deviates and second by using a Monte Carlo study to support the previously obtained theoretical results. (JEL C12, C15, C22)

1. Introduction

The process of correctly identifying the behavior under which an observed time series is generated, according to the Box and Jenkins (1976) methodology, can sometimes become a very challenging issue for many economic time series analysts. The truth is that, although it is believed that most macroeconomic time series are generated by a unit autoregressive root process, in practice it is not easy to detect whether a given time series is stationary or non-stationary given the existing testing procedures. In fact, recent empirical studies by Schwert (1989) and Agiakloglou and Newbold (1992) have reported simulation evidence showing that the performance of some common and widely used unit root test statistics is not satisfactory in moderately large samples even for the simplest possible model with one moving average term the ARIMA (O, 1, 1) process.

Bierens (1993), on the other hand, in a recent paper has developed a test statistic for testing for a unit autoregressive root based on the sample autocorre-

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lations of an observed time series. The test statistic for a given time series X_t, t= 1, 2, ..., n, is defined in the following way

$$R(\mu) = \frac{n[r_k - 1]}{k|\eta(k)|^{\mu}} \quad \text{if} \quad \eta(k) > 0 \qquad (1.1)$$

$$= -n^2 \quad \text{if} \quad \eta(k) \leq 0$$

where

$$\eta(k) = \frac{k(r_{k+1} - r_k)}{r_k - 1}$$

and r_k, the k^{th} sample autocorrelation, is defined as

$$r_k = \frac{C_k}{C_o} = \frac{\frac{1}{n}\sum_{t=k+1}^{n}(x_t - \bar{x})(x_{t-k} - \bar{x})}{\frac{1}{n}\sum_{t=1}^{n}(x_t - \bar{x})^2} \qquad (1.2)$$

for k = 1, 2, ..., χ is the sample mean and critical values for this test can be obtained from Bierens (1993).

This paper examines the performance of the (1.1) test statistic, proposed by Bierens (1993), for testing the null hypothesis for a unit autoregressive root based on the sample autocorrelations for series of one and three hundred observations generated by the simplest possible model with one moving average term the ARIMA (0,1, 1) process, first by applying the test to the means of the sample autocorrelations obtained as a ratio of two quadratic forms and second by using a Monte Carlo study.

Although Bierens (1993) has also developed a unit root test statistic similar to the one defined in (1.1) in which the sample autocorrelation r_k is defined as the OLS estimator of the coefficient *fa* in the auxiliary regression

$$(x_t - \bar{x}) = \beta_k (x_{t-k} - \bar{x}) + u_t \qquad (1.3)$$

where u_t is white noise and critical values for this test can be obtained from Fuller (1976), this test statistic however is not part of the objective of this paper for several reasons. First as reported by Bierens (1993) the performance of the unit root test based on the sample autocorrelations obtained from (1.3) is not as good as the one obtained from the sample autocorrelations defined in (1.2). Second the test based on the sample autocorrelations defined in (1.3) is simply an extension of the Dickey-Fuller test and therefore it can also be implemented through a regression analysis. Lastly and most importantly the sample autocorrelations obtained out of most computer packages are in the form of (1.2) and not in the form of (1.3).

2. The mean of the Sample Autocorrelations of the ARIMA (0, 1, 1) process

Let $\{X_t\}$ be a sequence of observations, t= 1, 2, ..., n, generated from an ARIMA (0, 1, 1) process, i.e.,

$$X_t - X_{t-1} = \varepsilon_t - \theta\varepsilon_{t-1} \tag{2.1}$$

where ε_t is white noise and $|\theta| < 1$. Without loss of generality assume that the process has mean zero and by recursively substituting X_{t-1} equation (2.1) becomes

$$X_t = \varepsilon_t + \lambda\varepsilon_{t-1} + \lambda\varepsilon_{t-2} + \ldots + \lambda\varepsilon_1 \tag{2.2}$$

where $\lambda = 1-\theta$. Hence, equation (2.2) can be written as $x = P\xi$ where x is a (nx1) vector of observations, P is an (nxn) matrix with unity on the main diagonal, zero and λ on the upper and lower triangular respectively and ξ is a (nx1) vector of independent standard normal deviates. Therefore the sample autocorrelation r_k of an ARIMA (0, 1, 1) process can be written as a ratio of two quadratic forms in the standard normal white noise process ξ_t

$$r_k = \frac{\xi' P' A Q_k A P \xi \,/\, (2n)}{\xi' P' A P \xi \,/ n} = \frac{\xi' F_k \xi / 2}{\xi' F_o \xi}$$

where A is a (nxn) idempotent matrix defined as $A = [I - 1\,1'/n]$ with 1 be a (nxl) vector of ones and Qk is a (nxn) symmetric matrix with unity on the k^{th} super and sub diagonals and zero elsewhere.

From Marriott and Pope (1954) and Kendall (1954) the mean of the sample autocorrelation rk can be approximately written as

$$E[r_k] = \frac{E[C_k]}{E[C_o]} \left[1 - \frac{E[C_k C_o]}{E[C_k]\,E[C_o]} + \frac{E[C_o^2]}{[E[C_o]]^2} \right] \qquad (2.3)$$

to order n^{-1}.

From Wichern (1973)

$$E[C_o] = \frac{1}{n} \text{ tr}(F_o)\,\sigma_\xi^2 \quad \text{and} \quad E[C_k] = \frac{1}{2n} \text{ tr}(F_k)\,\sigma_\xi^2$$

and from Kumar (1973)

$$E[C_k\, C_o] = \frac{1}{2n^2} \left[\text{tr}(F_k)\, \text{tr}(F_o) + 2\text{tr}(F_k F_o) \right] \sigma_\xi^4$$

and

$$E[C_o^2] = \frac{1}{n^2} \left[(\text{tr}(F_o))^2 + 2\text{tr}(F_o^2) \right] \sigma_\xi^4$$

where here $\sigma^2_\cdot = 1$, and therefore the right-hand side of equation (2.3) can be evaluated. Similar to this work has also be done and presented in a recent paper by Newbold and Agiakloglou (1993) to compute and evaluate the mean and the variance of the sample autocorrelations generated by ARFIMA (0, d, 0) models.

The (1.1) unit root test is then applied to the means of the first twenty sample autocorrelations, allowing μ to take values from 3 to 5, for series of 100 and 300 observations generated by ARIMA (0,1,1) models using values of 0= -0.9, -0.5, 0.0, 0.2, 0.5, and 0.8. Although the unit root hypothesis for n= 100 is typically rejected at various nominal level tests for all values of Θ, particularly for 0= 0.8 and 0.5 the unit root hypothesis is rejected always at the nominal 1% level test for all values o k and μ, for the negative values of θ including the random walk process the unit root hypothesis is not rejected only if low values of k are used. On the other hand, as the sample size increased the test did not improve its performance. For all positive values of Θ, except for θ= 0.2 using μ= 3 and k= 4-19 and μ= 4 and k= 6-14, the unit root hypothesis is still rejected at various nominal level tests, mentioning that for all cases of θ= 0.8 and for most cases of θ= 0.5 the null hypothesis is rejected at the nominal 1% level test, whereas for the non-positive values of θ the unit root hypothesis is not rejected almost for all values of μ and k.

Furthermore using the magnitude of the test statistic as a method of selecting the most appropriate values o k and μ for these models and sample sizes the following statements can be made. For the positive values of θ the magnitude of the test statistic suggests that it is better to use a high value of k for θ= 0.8 and some intermediate value of k for θ= 0.5 and 0.2. Using however the highest possible value of k, i.e., k= 19, the unit root hypothesis for 0= 0.8 is still rejected at the nominal 1% level test for both sample sizes. For the non-positive values of θ the magnitude of the test statistic indicates that it is better to use low values of k in which indeed in this case the unit root hypothesis is not rejected. Moreover, for all values of k and θ the magnitude of the test statistic increases in absolute terms as the value of μ increases implying that it is better to use low values of μ.

Recall that for series of 100 and 300 observations Bierens (1993) takes the values of k to be 12 and 14 respectively given that the value of μ is *a priori* chosen to be four although it is not clear whether or not these values of k and μ are the most appropriate values for these sample sizes and for these ARIMA models. In essence, this is the same problem that Schwert (1989), Agiakloglou and Newbold (1992), Hall (1994) and Ng and Perron (1995) have dealt with by investigating the possibility of determining the order of the approximating autoregression for given sample size in the process of examining the performance of the augmented Dickey-Fuller test.

3. The Performance of the Unit Root Test Based on a Monte Carlo Study

As an effort to investigate whether or not the theoretical results previously presented can be supported on empirical grounds, the (1.1) unit root test is next applied to the first twenty sample autocorrelations for series of 100 and 300 observations generated by ARIMA (0,1,1) models using the same values of the moving average parameter as in the previous section and the results of the performance of the test based in 1,000 replications are reported in Tables 1 and 2 respectively. The only difference here is that Tables 1 and 2 report for all values of μ those values of k in which the number of rejections of the unit root hypothesis is closer to the nominal 5% —and 10% if needed— level test in addition to proposed by Bierems values of k and μ for these sample sizes.

Perhaps the most remarkable feature of Tables 1 and 2 is the fact that the unit root hypothesis is rejected very often for all positive values of the moving average parameter regardless of the sample size. For example, for n= 100 the unit root hypothesis is rejected at the nominal 5% level test on 1,000 trials 599 times for 0= 0.8 using k= 18 and μ= 3 and 357 times for 0= 0.2 using k= 7 and μ= 3. The performance of the test is not satisfactory even for the random walk process. Using the smallest possible values of μ and k, i.e., μ= 3 and k= 1, the unit root hypothesis for n= 100 and 300 at the nominal 5% level test is rejected 177 and 95 times respectively out of 1,000 trials. The test seems to behave well, in the sense that the number of the rejections of the unit root hypothesis is approximately close to the nominal level tests, only for the negative values of the moving average parameter and only if low values of k are chosen regardless of the value of μ.

Moreover, there are four more interesting points that should be mentioned. First and most importantly it is fascinating to realize that the empirical significance levels of the test based on the proposed by Bierens values of k and μ for both sample sizes are extremely high for all values of the moving average parameter including the negative values of θ in which the performance of the test is satisfactory using low values of k. For example for n= 300 the unit root hypothesis for θ= 0.2 is rejected at the nominal 5% level test 354 times out of 1,000 trials using μ= 4 and k= 14. Therefore any test conclusions based on these values of k and μ would have produced misleading results. Second, the best chosen values of k reported in Tables 1 and 2 for all values of θ are those values of k that were expected to obtain based on the magnitude of the test statistic that was previously discussed. Moreover the number of rejections of the unit root hypothesis for all values of θ and for given value of k is smaller only when μ= 3. Third, it is also interesting to indicate how often the unit root hypothesis is rejected for both sample sizes especially for the large positive values of θ due to the fact that n(k) is negative. Lastly, the estimates of the means of the sample autocorrelations, although are not reported here but Tables are available upon request, are very similar to those estimates of the means obtaines through the (2.3) equation.

4. Summary

The performance of the unit root test based on the sample autocorrelations proposed by Bierens (1993) is investigated in this paper for series of one and three hundred observations generated by the simplest possible model with one moving average term the ARIMA (0,1,1) process. Unfortunately, as empirically shown in section 3 and supported on the grounds of using the means of the sample autocorrelations obtained from section 2, the performance of the unit root test is not satisfactory even when the sample size increases, that is also the reason why the power of the test in the absence of a unit autoregressive root is not examined, and that it is strongly affected not only by the values of k or μ but also by the values of the moving average parameter.

For all positive values of the moving average parameter even for the random walk process this study failed to find significance levels approximately close to the nominal level tests for series of 100 and 300 observations. On the other hand, the test seems to perform well for all negative values of the moving average parameter as long as low values of k are chosen. However if the proposed by Bierens (1993) values of k and μ are used for these sample sizes then the significance levels for all values of the moving average parameter will be far away from the nominal level tests.

TABLE 1

Number of rejections of the unit root hypothesis in 1,000 trials for series of 100 observations generated by ARIMA (0,1,1) models

θ	k	μ = 3	μ = 4	μ = 5	$\eta(k) \leq 0$
0.8	18	599 612	620 626	633 638	415
	12		672 688		415
0.5	15	498 524	545 569	581 599	158
	12		565 587		140
0.2	7	357 411	425 463	463 497	12
	12		490 526		41
0.0	1	177 238	224 278	268 316	0
	12		446 469		47
-0.5	2	36 56	46 67	59 73	0
	12		443 473		42
-0.9	2	27 40	32 45	38 50	0
	12		404 439		36

Numbers in each cell are for nominal 5% and 10% level tests respectively.

TABLE 2

Number of rejections of the unit root hypothesis
in 1,000 trials for series of 300 observations
generated by ARIMA (0,1,1) models

θ	k	μ 3	μ 4	μ 5	η (k) ≤ 0
0.8	19	651 669	663 680	675 694	335
	14		759 770		329
0.5	17	429 477	489 532	539 572	30
	14		537 576		19
0.2	10	267 308	315 369	366 403	0
	14		354 403		3
0.0	1	95 141	115 165	142 190	0
	14		254 301		1
-0.5	4	52 76	69 92	80 107	0
	14		266 296		1
-0.9	4	40 64	49 76	59 86	0
	14		266 309		0

Numbers in each cell are for nominal 5% and 10% level tests respectively.

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