OPTIMAL INVESTMENT IN EDUCATION: A TWO-SECTOR GROWTH MODEL *

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I. INTRODUCTION

It has been increasingly recognized in the economic literature that human skills are among the important inputs into production and that economic development and growth should be accompanied by an expansion in education.

Education, as a process of human capital production, contributes directly to economic growth by increasing the productivity of the labour force, let alone the additional socially desirable effects it may have.

In the following pages, a dynamic model is presented in which education is explicitly taken into account in the form of endogenous labour-augmenting technological progress. The model is formally stated in section II. The solution, along with the analysis of its basic properties, is worked out in section III. The optimal path of the crucial economic (state) variables, derived from the mathematical solution of the model, is synthesized in Section IV. Finally, the conclusion is contained in Section V.

I. THE MODEL

We assume that output is given as a function of physical capital, and quality-corrected labour force E.N., where N is the number of workers and E is a labour-augmenting quality multiplier (with a 'per worker' dimension).

So, if Y represents output, we assume :[1]

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*This article is largely based on my M.Sc. dissertation submitted to the University of Bristol in September 1975.

1. Y,K,E,N are all functions of time, but the argument t has been omitted for simplicity.

$$Y = F(K, E \cdot N)^{2} \qquad (1)$$

This aggregative production function, which summarizes the technically efficient possibilities for production of output from capital and (quality-corrected) labour, is further assumed invariant over time and twice differentiable with : Positive but diminishing marginal products i.e

$$\frac{\partial F}{\partial K} > 0, \quad \frac{\partial^2 F}{\partial K^2} < 0$$

$$\frac{\partial F}{\partial (E \cdot N)} > 0, \quad \frac{\partial^2 F}{\partial (EN)^2} < 0 \qquad (2)$$

Also : $$\lim_{K \to 0} \frac{\partial F}{\partial K} = \infty, \quad \lim_{R \to \infty} \frac{\partial F}{\partial K} = 0$$

$$\lim_{(E \cdot N) \to 0} \frac{\partial F}{\partial (EN)} = \infty, \quad \lim_{(E \cdot N) \to \infty} \frac{\partial F}{\partial (EN)} = 0 \qquad (3)$$

So that both marginal products start at infinity and diminish to zero.[3]

It is also assumed that the production function exhibits constant returns to scale, so, for any positive scale factor a :

$$F(\alpha K, \alpha EN) = \alpha F(K, EN) = \alpha Y \qquad (4)$$

Using property (4) we can now write (1) as follows :

$$Y = F(K, N) = NF\left(\frac{K}{N}, E\right) = NF(\kappa, E) \quad \left(\text{with } \kappa = \frac{K}{N}\right) =$$

$$= NEF(\kappa/E, 1) = NEf(\kappa/E) \quad (\text{where } f(\cdot) = F(\cdot, \cdot)) \qquad (5)$$

There are two categories of nonconsumption (investment) use of the output flow. The first is allocation of part of output to accumulation of further capital stock. Let I be the rate of flow of output directed toward such additions to capital; I is then the rate of gross investment in physical capital. Assuming that

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2. The idea for this production function is due to Griliches. See : Zvi Griliches «Notes on the role of education in production functions and growth accounting» in W.L. Hansen (ed.) «Education. Income, and Human Capital». National Bureau of Economic Research, New York 1970.

3. The assumptions on the productivity of the 'effective labour' can be perhaps supported empirically. Private returns from elementary education apparently exceed those from high school education, which in turn exceed those from college education. One has positive returns from continuing his education, of course, but they may diminish.

the existing capital stock depreciates at the constant proportionate rate δ_1, the investment needed to maintain the capital stock at existing levels is $\delta_1 K$ so, the gross investment identity (in physical capital) states that :

$$I = \dot{K} + \delta_1 K \qquad (6)$$

A second category of expenditure arises because it is assumed that workers are trained (educated).

Let I_E be the (gross) investment in the education sector. Assuming that 'skills' depreciate at the proportionate rate δ_2, the second investment identity states that :

$$\frac{I_E}{N} = \dot{E} + \delta_2 E \qquad (7)$$

where $\dot{E}$ represents net additions to -the 'stocks' of skills. Since E is expressed in per worker terms, investment is required to be divided by N.

Finally, the population (and hence the labour force, since full employment is always assumed to prevail) is taken to grow at the given, exogenously determined exponential rate η :

$$\dot{N} = \eta N \qquad (8)$$

Consumption is given as the residual :

$$C = F(K,EN) - I - I_E = F(K,EN) - (\dot{K} + \delta_1 K) - N(\dot{E} + \delta_2 E). \qquad (9)$$

This equation thus reflects the fact that the total output flow Y is allocated among three uses. A flow C for present consumption, a flow I representing the consumption foregone because of the use of resources in adding to stocks of physical capital, and a flow I_E representing the consumption foregone in order to devote resources to the creation of 'human capital' (skills).

Let us now introduce new variables as follows :

$$s_1 = \frac{I}{Y}, \qquad s_2 = \frac{I_E}{Y}$$

so that equations (6), (7), (9) become :

$$\dot{K} = s_1 Y - \delta_1 K \qquad (10)$$

$$\dot{E} = \frac{s_2 Y}{N} - \delta_2 E \tag{11}$$

$$C = (1 - s_1 - s_2)\,Y \tag{12}$$

respectively. Now, from (10), we have :

$$\frac{\dot{K}}{N} - \frac{s_1 Y}{N} - \delta_1 \frac{K}{N} = (\text{using } (5))$$

$$= \frac{s_1 NEf(\kappa/E)}{N} - \delta_1 \kappa = s_1 Ef(\kappa/E) - \delta_1 \kappa \tag{13}$$

$$\text{But,, } \dot{\kappa} = \frac{d}{dt}\left(\frac{K}{N}\right) = \frac{N\dot{K} - K\dot{N}}{NN} = \frac{\dot{K}}{N} - \kappa \frac{\dot{N}}{N} \tag{14}$$

Combining (13, (14), (8), we get :

$$\dot{\kappa} = s_1 Ef(\kappa/E) - (\delta_1 + \eta)\kappa \tag{15}$$

From (11) we get immediately :

$$\dot{E} = \frac{s_2 NEf(\kappa/E)}{N} - \delta_2 E = s_2 Ef(\kappa/E) - \delta_2 E$$

$$= (s_2 f(\kappa/E) - \delta_2)E = \dot{E} \tag{16}$$

The economic objective of the planner is assumed to be based on standards of living as measured by consumption per worker [4]. In particular, it is assumed that the central planner has a utility function, giving utility, U, at any instant of time, as a function of consumption per worker at that time.

$$U = U(c(t)).$$

The utility function is assumed twice differentiable with positive but diminishing marginal utility for all positive levels of consumption per worker :

$$U'(e) > 0, \qquad U''(c) < 0 \qquad \text{all } c \in (0, \infty)$$

4. See : Michael D. Intriligator «Mathematical Optimization and Economic Theory» Prentice-Hall, Inc. 1971, page 408.

so that U(.) is a strictly concave [5], monotonic increasing [6] function.

We also assume that :

$$\lim_{c \to 0} U(c) = -\infty \;^{7}, \qquad \lim_{c \to 0} U'(c) = +\infty .$$

The problem now facing the planner is to choose $s_1(t)$ and $s_2(t)$ so as to maximize the discounted sum of the future utilities of per capita consumption, where :

$$\text{per capita consumption} = (1 - s_1 - s_2)\frac{Y}{N} = (1 - s_1 - s_2)\,\frac{NEf(\kappa/E)}{N} =$$

$$= (1 - s_1 - s_2)Ef(\kappa/E) \qquad (17)$$

Formally :

$$\underset{\{s_1,\, s_2\}}{\text{maximize}} \quad T = \int_0^{\infty} e^{-\rho t}\{U\,[(1 - s_1 - s_2)\,Ef(\kappa/E)]\}\,dt$$

subject to :

$$\dot{\kappa} = s_1 Ef - \lambda\kappa \;\; (\lambda = \delta_1 + \eta) \qquad \kappa(o) = \kappa_0, \text{ given}$$

$$\dot{E} = (s_2 f - \delta_2)E \qquad E(o) = E_0, \text{ given} \qquad (18)$$

$$s_1 + s_2 \leqslant 1$$

$$s_1,\ s_2 \geqslant 0 \;^{8}$$

where P ($>$ o)is the social rate of discount or time preference. To solve this problem, we apply the Maximum Principle of Pontryagin [9]. Thus, we form the Hamiltonian :

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5. This means that we attribute greater weight to the marginal utility of per capita consumption of a 'poor' generation than that of a 'rich' one.

6. We rule out saturation.

7. The consumers are 'infinitely unhappy' with zero consumption.

8. This assumption, implying the irreversibility of investment, deserves a comment. The non-negativity of gross investment implies that capital, once installed in a sector, cannot be used in the other. In the context of the present model it means, heuristically speaking, that «factories cannot be used as schools and vice-versa». For a detailed analysis of the irreversible investment case in the one-sector model, see : Arrow «Optimal Capital Policy with Irreversible Investment» in Value, Capital and Growth, papers in honour of Sir John Hicks, J.N. Wolf (ed.) Edinburgh University Press, 1968.

9. Pontryagin, L.S., V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko : The Mathematical Theory of Optimal Processes, New York : Wiley, 1962.

$$H = e^{-\rho t}\{U[(1 - s_1 - s_2)Ef] + P_1(s_1Ef - \lambda\kappa) + P_2(s_2f - \delta_2)E,\} \qquad (19)$$

Then, for any optimal path k(t), E(t), there must exist[10] continuous auxiliary functions $P_1(t)$, $P_2(t)$, for which the following conditions hold :

(i)
$$\dot{\kappa} = \frac{\partial}{\partial P_1}(He^{\rho t}) = s_1 Ef - \lambda\kappa \qquad (20)$$

$$\dot{E} = \frac{\partial}{\partial P_2}(e^{\rho t}) = (s_2 f - \delta_2)E \qquad (21)$$

(ii)
$$\frac{d}{dt}(e^{-\rho t}P_1) = -\frac{\partial H}{\partial \kappa} \Rightarrow \dot{P}_1 = (\rho + \lambda)P_1 - qf' \qquad (22)$$

$$\frac{d}{dt}(e^{-\rho t}P_2) = -\frac{\partial H}{\partial E} \Rightarrow \dot{P}_2 = (\rho + \delta_2)P_2 - q\left(f - \frac{\kappa}{E}f'\right) \qquad (23)$$

$$\text{where } q = (1 - s_1 - s_2)\,U'(\cdot) + P_1 s_1 + P_2 s_2 \qquad (24)$$

(iii) s_1, s_2, are piecewise continuous functions of time, maximizing H for all t, and :

(iv) The transversality condition :

$$\lim_{t\to\infty} P_1 e^{-\rho t} = 0, \qquad \lim_{t\to\infty} P_2 e^{-\rho t} = 0. \qquad (25$$

$P_1(t)$ has the interpretation of the social demand price of investment (in physical capital) at time t (in terms of utility of consumption foregone at time t). Thus, the differential equation (22) has the following interpretation : In an economy in which capital rental is rewarded by its marginal value product, the price of a unit of capital must change so as to reward the renter for «waiting», less the value of net rentals received[11].

$P_2(t)$ is interpreted in a similar manner.

It is now obvious that the Hamiltonian represents the value (in terms of utility) of per capita output, discounted to time zero.

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10. Sufficiency of the conditions is ensured by the concavity of the utility function.

11. See K. Sheel «Applications of Pontryagin's Maximum Principle to Economics», in «Mathematical Systems. Theory and Economics», ed. H. W. Hahn and G.P. Szego, Berlin : Springer-Zertag, 1969.

II. PATTERNS OF MOTION

By condition (iii), we consider the following five phases (see figure 1).

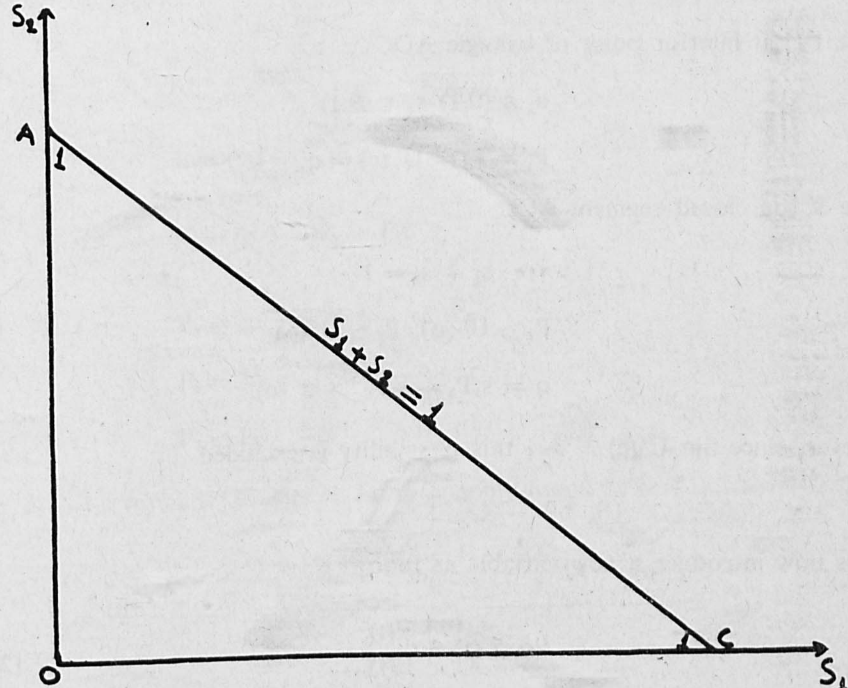

Figure 1.

Phase I (on open segment AO).

$$s_1 = o, \ s_2 \ \varepsilon \ (0,1)$$

$$P_1 \leqslant P_2 = U' \ (c)$$

$$q = U' \ (c)$$

Phase II (on open segment OC).

$$s_2 = o, \ s_1 \ \varepsilon \ (0,1)$$

$$P_2 \leqslant P_1 = U' \ (c)$$

$$q = U' \ (c)$$

Phase III (at point 0).

$$s_1 = s_2 = o$$

$$P_1 \leqslant U'(c),\ P_2 \leqslant U'(c)$$

$$q = U'(c)$$

Phase IV (at interior point of triangle AOC).

$$s_1\ \varepsilon\ (0,1)\ s_2\ \varepsilon\ (0,1)$$

$$P_1 = P_2 = U'(c) = q$$

Phase V (on closed segment AC).

$$s_1 + s_2 = 1$$

$$P_1 \geqslant U'(o),\ P_2 \geqslant U'(o)$$

$$q = s_1 P_1 + s_2 P_2 > U'(o).$$

However, since $\lim_{c \to o} U'(c) = \infty$, this possibility is excluded.

Let us now introduce a new variable as follows :

$$x(t) = \frac{\kappa(t)}{E(t)} \quad (26)$$

Then, $\dot{x} = \dfrac{E\dot{\kappa} - \kappa\dot{E}}{EE} = \dfrac{\dot{\kappa}}{E} - x\,\dfrac{\dot{E}}{E}$. Combining this with relations (15) and (16, we get :

$$\dot{x} = -x\,(s_2 f + \lambda - \delta_2) + s_1 f \quad (27)$$

where now f (·) is a function of x.

In order to be able to derive explicit solutions we will further assume that our utility function belongs to the constant elasticity class, with :

$$\sigma = \frac{U''(c)}{U'(c)} \cdot c = \text{the (negative) elasticity of marginal utility (with respect to consumption).} \quad (28)$$

We are now ready to examine the motion generated by each phase in turn.

Phase I

The economy decumulates (physical) capital to the limit. Equations (20) — (23) become:

(20.1) $\dot{\kappa} = -\lambda\kappa$

(21.1) $\dot{E} = (s_2 f - \delta_2)F$

(22.1) $\dot{P}_1 = (\rho + \lambda)P_1 - P_2 f'$

(23.1) $\dot{P}_2 = (\rho + \delta_2 - f + xf')P_2$

$P_2 = U'(c) = q$

$P_1 \leqslant P_2$

From $P_2 = U'(c)$ and (20.1), (21.1), s_2 is easily found to be:

$$s_2 = \frac{-\dfrac{\rho + \delta_2}{\sigma} + \delta_2 + \dfrac{f - xf'}{\sigma} + \dfrac{xf'}{f}(\lambda - \delta_2)}{f - xf'} \qquad (29)$$

Equation (27) becomes:

$$\dot{x} = -x(s_2 f + \lambda - \delta_2) \qquad (30)$$

where the value of s_2 is given by (29).

In this phase, k is monotonically decreasing to zero, but E will rise or fall, depending on whether $s_2 f$ exceeds or falls short of δ_2.

The trajectories may roughly[12] be shown in the (k,E) plane in Figure 2.

Phase II

The economy decumulates skills to the limit[13]. Equations (20) - (23) now become:

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12. 'Roughly' because we do not know the exact convergence properties of equation (21.1). The same remark applies to the figure of Phase II.

13. This does not contradict the assumption of full-employment. It is only the quality (productivity) of the labour force which is affected (i.e. reduced).

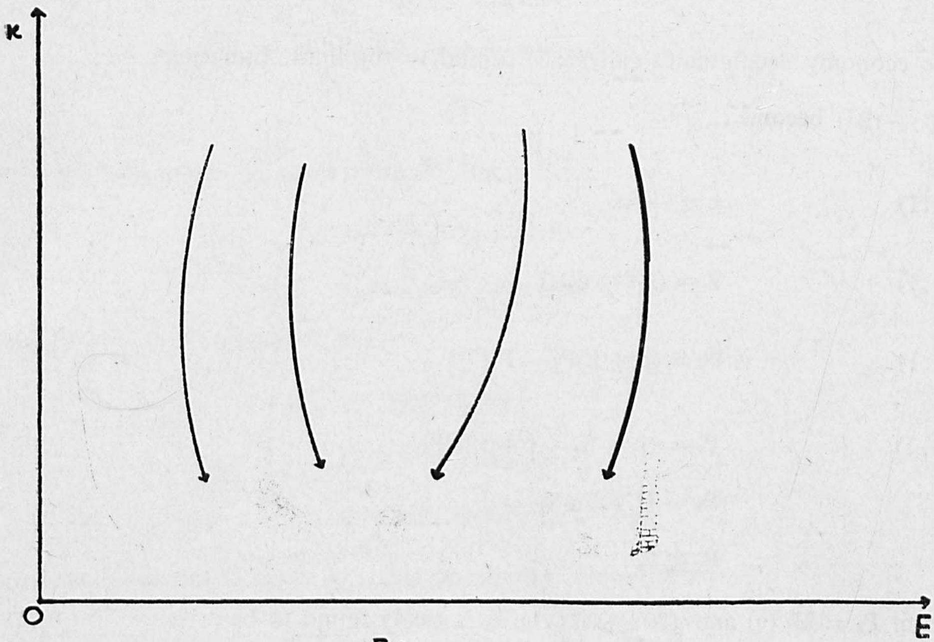

Figure 2.

(20.2) $$\dot{\kappa} = s_1 E f - \lambda \kappa$$

(21.2) $$\dot{E} = - \delta_2 E$$

(22.2) $$\dot{P}_1 = (\rho + \lambda - f') P_1$$

(23.2) $$\dot{P}_2 = (\rho + \delta_2 - P_2 - P_1 (f - xf\ )$$

$$P_1 = U'(c) = q$$

$$P_2 \leqslant P_1$$

Equation (27) becomes :

$$\dot{x} = x\,(\lambda - \delta_2) + s_1 f. \tag{31}$$

The value of s_1 is found from the relations : (20.2), (21.2) and $P_1 = U'(c)$:

$$s_1 = \frac{\delta_2 - \dfrac{\rho + \lambda - f'}{\sigma} - \dfrac{xf'}{f}(\delta_2 - \lambda\ )}{f'} \tag{32}$$

The trajectories for phase II are shown in figure 3.

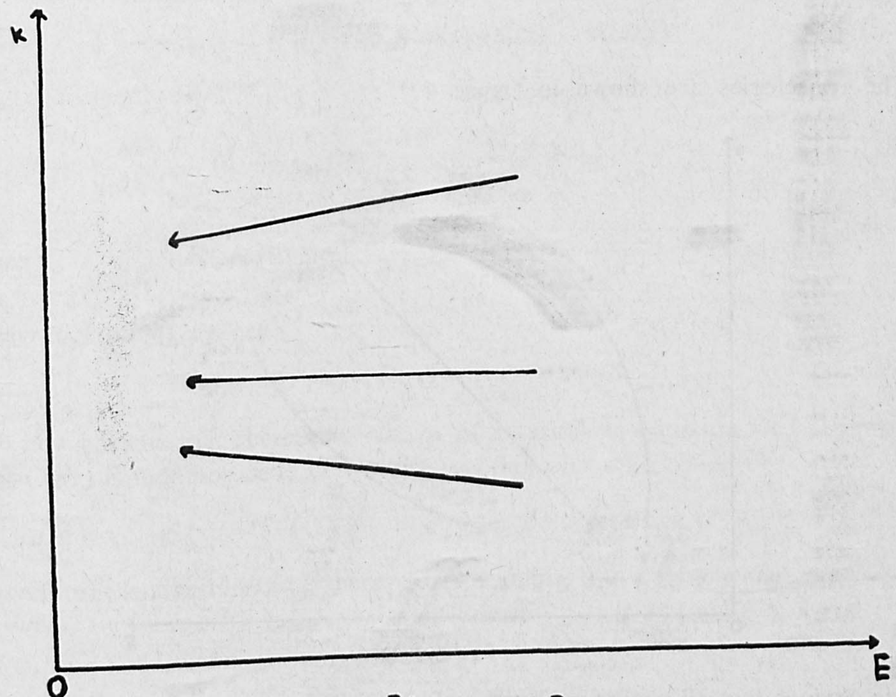

Figure 3.

Phase III

Here, both k and E decrease monotonically and all output is devoted to consumption. Equations (20) - (23) become :

(20.3) $$\dot{\kappa} = -\lambda\kappa$$

(21.3) $$\dot{E} = -\delta_2 E$$

(22.3) $$\dot{P}_1 = (\rho + \lambda) P_1 - qf'$$

(23.3) $$\dot{P}_2 = (\rho + \delta_2) P_2 - q(f - xf')$$

$$P_1 \leqslant U'(c) = q$$

$$P_2 \leqslant U'(c).$$

As regards equation (27) it now takes the form :

$$\dot{x} = x(\lambda - \delta_2) \tag{33}$$

The trajectories are shown in figure 4 [14].

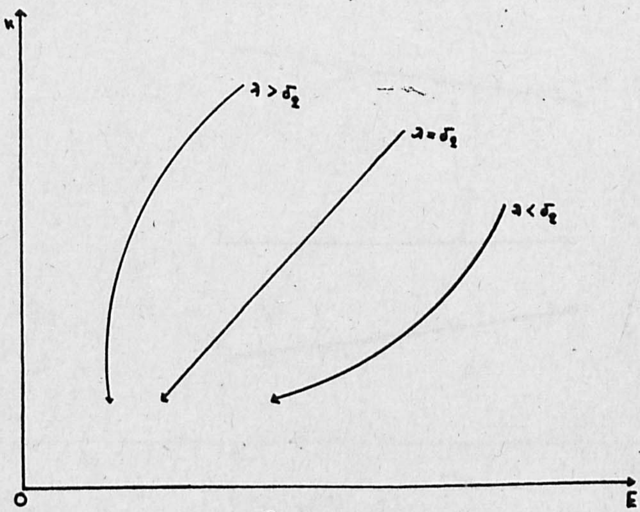

Figure 4.

It is obvious that all phases analysed so far cannot be optimal for ever, since they all lead to zero consumption. Therefore, the final phase has to be phase IV, to which we now turn.

Phase IV.

Equations (20) - (23) become :

$$\text{(20.4)} \qquad \dot{\kappa} = s_1 E f - \lambda \kappa$$

$$\text{(21.4)} \qquad \dot{E} = (s_2 f - \delta_2) E$$

$$\text{(22.4)} \qquad \dot{P}_1 = (\rho + \lambda) P_1 - q f'$$

$$\text{(23.4)} \qquad \dot{P}_2 = (\rho + \delta_2) P_2 - q(f - x f')$$

$$P_1 = P_2 - U'(c) = q.$$

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14. $\lambda = \delta_2$: Both κ and E approach zero at the same speed (x remains constant)
$\lambda > \delta_2$: k decreases 'faster' than E (x increases)
$\lambda < \delta_2$: E » » » k (x decreases).

In order to remain in phase IV we should also have $\dot{P}_1 = \dot{P}_2$ which implies :

$$\rho + \lambda - f'(x) = (\rho + \delta_2) - (f(x) - xf'(x)) \qquad (34)$$

or, equivalently :

$$\frac{\partial F}{\partial K} - (\rho + \lambda) = \frac{\partial F}{\partial (E \cdot N)} - (\rho + \delta_2) \qquad (35)$$

since : $\frac{\partial F}{\partial K} = f'(\kappa/E)$ and $\frac{\partial F}{\partial (EN)} = f(\kappa/E) - \frac{\kappa}{E} f'(\kappa/E)$.

Rearranging (34) we get :

$$g(x) \equiv f'(x)(1 + x) - f(x) = \lambda - \delta_2 \qquad (36)$$

We now assume that there exist values of x satisfying equation (36).
Then, g(x) is monotonically decreasing function of x with :

$$g'(x) = f''(x)(1 + x) < 0 \text{ for every positive } x.$$

Therefore it is invertible and there exists a unique $x = x^*$ such that : $x^* = g^{-1}(\lambda - \delta_2)$.

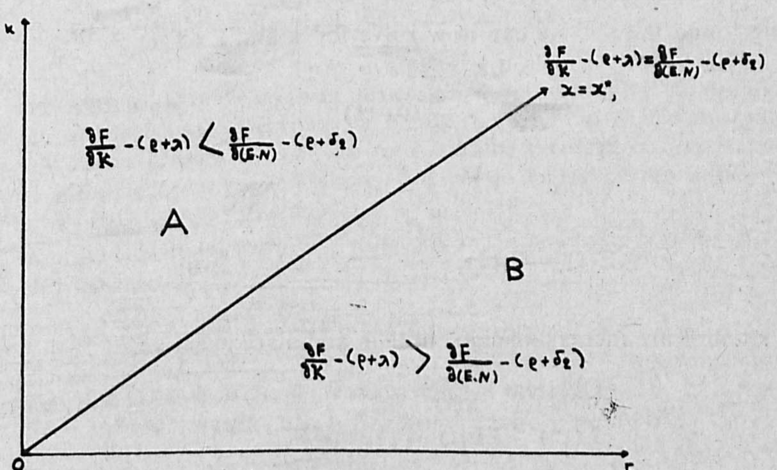

Figure 5.

So in phase IV, both k and E change in such a way that their ratio remains constant and equal to $\times$ * (see figure 5). Along the ray $0\times$ * the net rate of return on capital equals the net rate of return on skills. (Therefore, $\times$ * is the 'modified golden rule' capital-(skilled) labour ratio.)

Let us call $\rho + \lambda - f'(x^*) = \omega$

and assume it is negative to make sure that consumption grows rather than decays along the optimal path.
We are in a position now to solve explicitly and find the values of all the relevant variables in phase IV.
Equation (27) gives :

$$o = s_1 f(x^*) - x^*(\lambda + s_2 f(x^*) - \delta_2). \tag{37}$$

Assuming that phase IV is entered at time t^*, equations (22.4) and (23.4) can be solved immediately for any $t > t^*$ to give :

$$P_1\ (t) = P_2\ (t) = P(t^*)_e\ \omega (t - t^*). \tag{38}$$

As t tends to infinity, P_i tends to zero so that the transversality condition is satisfied.
Using (20.4), (21.4), (37) and $P_1 = P_2 = U'(c)$ we can solve for s_1 and s_2 to get :

$$s_1 = \frac{x^*(\frac{\omega}{\sigma} + \lambda)}{f}\ ,\ s_2 = \frac{\frac{\omega}{\sigma} + \delta_2}{f}\ . \tag{39}$$

Obviously, we have to assume that the values of s_1 and s_2, so found, are such that the constraint $s_1 + s_2 < 1$ is not violated,
Having found the s's, we can now solve for k and E to get :

$$\kappa(t) = \kappa(t^*)\, e^{\frac{\omega}{\sigma}(t - t^*)}\ ,\quad t \geqslant t^* \tag{40}$$

$$E(t) = E(t^*) e^{\frac{\omega}{\sigma}(t - t^*)}\ ,\quad t \geqslant t^* \tag{41}$$

Both k and E are increasing since both ω and s are negative.

$$\text{Also,} \frac{\kappa(t^*)}{E(t^*)} = \frac{\kappa(t)}{E(t)} = x^* \quad \text{for every } t \geqslant t^*.$$

It should be noted that although the mathematics is a little cumbersome, the results are very plausible on economic grounds. In phase IV, our balanced growth-equilibrium phase, the rate of investment in each sector is set so as to equalize the demand prices-P_1 and P_2 — and the supply price or opportunity cost $U'(c)$.

So long as this equality is preserved, there is no economic reason for altering the composition of investment (as well as its relation to consumption) and s_1, s_2 remain constant. Both κ and E rise at the same exponential rate, so that their ratio remains always constant. Consumption also rises at the same rate, but this makes (due to the concavity of the utility function) marginal utility fall. Demand prices P_1 and P_2 also fall, remaining equal to U'(c). This goes on until all three prices fal simultaneously to zero[15].

Alternatively, if the initial endowments on κ and E are not at the 'right' (turnpike) proportion, we simply have 'too much' of the one, or 'too little' of the other. Investment then is specialized into the one sector (phase I or II), or even none (phase III). The demand price in the sector where investment does not occur is below the supply price U'(c) and that is what makes investment unprofitable and hence undesirable.

Finally, phase IV is reached and then on the paths remain there for ever, as explained above[16].

III. SYNTHESIS OF THE OPTIMAL PATH[17].

All paths terminate in Phase IV where $P_1 = P_2$. Therefore, in order for the optimal path to switch from Phase I (where $P_1 \leqslant P_2$) to Phase IV,

$$\frac{\dot{P}_1}{P_1} \geqslant \frac{\dot{P}_2}{P_2} \tag{42}$$

should hold in the neighbourhood of the transition point.

15. This is the trick' of the transversality condition. The «end of the world»-infinity-exists mathematically, although it will never come. But if it ever came, it would find us very well prepared so that we would have nothing to lose. The transversality condition ensures that the prices become, zero at the right time, or, equivalently, that they do not become zero at the wrong time (i.e. prior to infinity).

16. In order to make sure that our criterion of optimality is meaningful, we have to assume that the integral :

$$\int_0^\infty e^{-\rho t}\, U[c\,(t)\,]dt, \text{ converges}$$

Since our final phase is phase IV the above requirement is equivalent to :

$$\int_{t^*}^\infty e^{-\rho t} U[c^*(t)]\, dt < \infty$$

where c* is consumption along phase IV. This integral now depends on the parameters ρ, ω, σ, so there can always be found suitable relations between them, ensuring the convergence of the integral.

17. The analysis in this section is based on : Koishi Hamada, «Optimal Capital Accumulation by an Economy facing an International Capital Market». Journal of Political Economy 1969.

See also : Ryder, «Optimal Accumulation in a Two-Sector Neoclassical Economy with Non-shiftable Capital». Journal of Political Economy, 77,No. 4, Part II(July-August 1969) 665-683.

But from (22.1), (23.1) in Phase I we have :

$$\frac{\dot{P}_1}{P_1} - \frac{\dot{P}_2}{P_2} = (\rho + \lambda) - \frac{P_2}{P_1} f' - (\rho + \delta_2) + (f - xf') \leqslant (\rho + \lambda) - f' - [(\rho + \delta_2) - (f' - xf')]$$

since $P_1 \leqslant P_2$.

Therefore (42) is impossible unless

$$(\rho + \lambda) - f' - [(\rho + \delta_2) - (f - xf)] \geqslant 0$$

which implies :

$$f' - (\rho + \lambda) \leqslant f - xf' - (\rho + \delta_2) \quad \text{or}$$

$$\frac{\partial F}{\partial K} - (\rho + \lambda) \leqslant \frac{\partial F}{\partial (E \cdot N)} - (\rho + \delta_2).$$

This shows that the transition from phase I to IV can occur only from the region northwest of the ray ox* in figure 5 (region A). Similarly, the transition from phase II to IV can occur only from region B, southeast of ox*.

Now, suppose the switch from phase I to II occurs. At the switching point we would have :

$P_1 = P_2$ and $\dot{P}_1 > \dot{P}_2$. This from (22.1) and (23.1), implies that :

$$\dot{P}_1 - \dot{P} = (\rho + \lambda) - f' - (\rho + \delta_2) + (f - xf').$$

Therefore, the switch from phase I to II should occur in the region A. Similarly, the switch from phase II to I should occur in the region B. However, the optimal trajectory must finally switch into phase IV from I and from region A.

So, if there were a switch to phase I from II within B, the trajectory then would have to :

a) either switch to phase IV from there-impossible, because the I to IV switch occurs in the A region,

b) or switch back to II within B and then to IV - again impossible, since the switch from I to II can occur only from the region A.

This contradiction implies that there cannot occur a switch from I to II and vice versa.

From each point in the optimal path in phase IV which lies on the ray ox*, we can find backward solutions in phase I which move in the region A [18]. Similarly,

18. It should be noted here that we do not actually know the values of both k and E in phase IV. What we known (and what 'matters') is their ratio (in the (x,t) plane the ray ox* of figure 5 would be represented by a single point). Also, we do not know the value of the shadow prices

from each point on ox *, we can find backward solutions in phase II which move in the region B.

As we prolong the backward solutions in phase I or II, we may eventually violate the conditions on control variables. For the solutions in phase I there may be a frontier beyond which the solutions do not give the value of s_2 satisfying $s_2 > 0$; for the solutions in phase II, a frontier beyond which they cease to give $s_1 > 0$.

This situation is roughly depicted in figure 6.

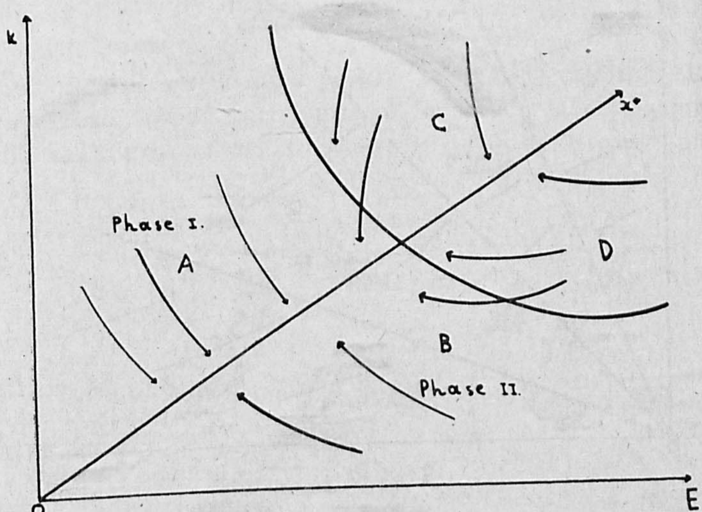

Figure 6.

In regions C and D, phases I and II cannot give meaningful solutions since capital as well as skills are excessive. Therefore, the only solution is to decumulate both stocks as fast as possible, that is, the solutions in phase III.

An interesting case may occur when $\lambda = \delta_2$. In this case, the differential equation in x has the following form :

$$\dot{x} = -x\ s_2 f \quad \text{in phase I} \tag{43}$$

$$\dot{x} = s_1 f \quad \text{in phase II} \tag{44}$$

$$\dot{x} = 0 \quad \text{in phase III.} \tag{45}$$

Clearly, (45) implies that x remains constant along phase III. Both k and E depreciate at the same rate and consequently their ratio never changes. P_1, P_2,

P(t*) at the time of entry into phase IV. It depends on both initial values k(o) and E(o) and not just their ratio. So, in order to find P(t*) we would have to solve first the 'quantity' system. This is why we cannot (as we might wish to) suggest a 'decentralization' procedure and find the initial prices, the optimal path then being determined from the solution of the 'dual' (pricing) problem, from the (then) given initial prices.

being functions of x alone, will change in a constant manner, and phase III, once entered, will never be departed from. Both κ and E will tend asymptotically to zero and phase IV will never be reached.

This situation is shown in figure 7.

If the initial point lies in regions C or D in figure 7, the optimal phase IV path is never attained.

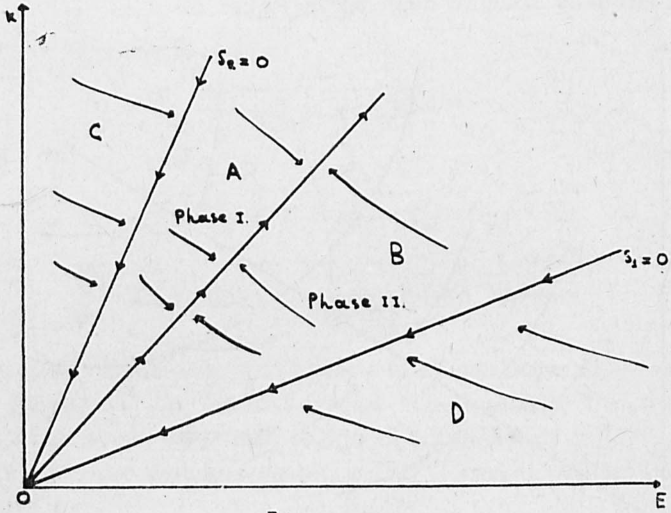

Figure 7.

We have, therefore, arrived at the following general description of the optimal path :

a) If the initial value of x lies on ox* (figure 6), then the path remains in phase IV all the time, travelling upward along the ray ox*.

b) If x(o) lies in region A, the path switches from phase I to phase IV.

c) If x(o) lies in B, the path switches from phase II to IV.

d) If the path lies in C (provided C exists), it switches from III to IV or from III to I to IV, or to zero (the case of figure 7).

e) If x (o) lies in D (again provided D exists), the path switches from III to IV or to II to IV or to zero.

IV. CONCLUSION AND EXTENSIONS.

Lack of skilled labour is a major problem of many developing countries (Iran, we think, is a characteristic example). In such a situation, the development planner would have to take explicit account of the necessity for education and training. In the context of the model presented and analysed above, he (the planner) should initially specialize investment into the educational sector until the labour force becomes 'sufficiently' skilled (productive) to run the capital goods industry. In the way education is treated in the model however, it 'matters' only to the extent that it contributes to the growth of output. This, it might be suggested, is a rather narrow view of the benefical effects of education. Education, after all, is a 'good thing' in itself. This consideration leads us to the following modification of the model[19].

The utility function may take the form :

$$U = U(C,E)$$

with the 'flow' of education having a positive effect on welfare. Also, we may introduce an explicit "production function' for skills, thus making the model more realistic, at the expense (as is usually the case) of greater mathematical complexity.

19. This suggestion, as well as the next one, was made to me by Professor J. Sandee.

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