A METHOD FOR DETERMINING THE UNIQUENESS OF THE INTERNAL RATE OF RETURN FOR AN INVESTMENT

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1. Introduction

Much of the present Capital Investment literature [1, 5] has been concerned with questions involving the uniqueness or nonuniqueness of the internal rate of return for a proposed investment. In this paper a new method for determining the uniqueness of the internal rate of return for investment is presented.

Also, an example is given, caltulating the number of rates of return for an investment project, using this new method.

2. Definitions and Notation

A few definitions are given before the methodology.

2.1 Given any sequence of real numbers $a_0, a_1, \dots, a_m, \dots$, it is said that there is a variation in sign at a_m provided either that $a_{m-1} a_m < 0$ or that $a_i a_m < 0$ and $a_{i+1} = a_{i+2} = \dots = a_{m-1} = 0$.

2.2 A sequence of real polynomials

$f(x), f_1(x), \dots, f_m(x)$

will be said to form a Sturm sequence for f(x), on the interval [a, b] in case the following are true.

(i) No two consecutive polynomials in the sequence vanish simultaneously on the interval.

(ii) If $f_j(r) = 0$ for $j < m$, then $f_{j-1}(r) f_{j+1}(r) < 0$

(iii) Throughout the interval [a, b], $f_m(x) \neq 0$, and

(iv) If $f(r)=0$, then $f'(r) f_1(x) > 0$.

2.3 By $z^{1/2}(r_1, r_2, \ldots, r_n)$, the following determinant of order n, is defined.

$$z^{1/2}(r_1, r_2, \ldots, r_n) = \delta \quad \begin{matrix} 1 & 1 \ldots & 1 \\ r_1 & r_2 \ldots & r_n \\ r_1^2 & r_2^2 \ldots & r_n^2 \\ \ldots & \ldots\ldots\ldots & \ldots \\ r_1^{n-1} & r_2^{n-1} \ldots & r_n^{n-1} \end{matrix}$$

Evidently, $z(r_1, r_2, \ldots, r_n)$, being the square of the determinant above, is symmetric and is a polynomial in the r_j ;

2.4 The power sums of the r_i, are defined as

$$s_v = \sum_{i=1}^{n} r_i^v.$$

3. Internal Rate of Return.

The internal rate of return is defined as the interest rate that reduces the present worth amount of a series of receipts and disbursements to zero. That is, the rate of return for investment proposal is the interest rate i* that satisfies the equation

$$P(i^*) = \sum_{j=0}^{n} a_j (1+i^*)^{-J} = 0 \tag{1}$$

where the investment proposal has a life of n periods and each of the real numbers a_j (j= 0, 1, 2, . . ., n) represents the net cash flow generated by the investment in j^{-th} period of its life.

If we let $x=(1+i)^{-1}$, which is strictly positive for i in the interval $[0, \infty]$, the present value function can be written as the following polybomial of degree n in x.

$$P(i) = \sum_{j=0}^{n} a\,(1+i)^{-j}$$

$$= \sum_{j=0}^{n} a_j x^j \tag{2}$$

Furthermore, the rate of return from a project whose annual cash flows are given by the sequence $a_0, a_1, \ldots, a_n$ is said to be unique, if there exists a unique real value of $i^*=i_0^*$ in $0 < i^* < \infty$ satisfying the equation (1).

From equation (1):

$$\sum_{j=0}^{n} a_j x^j = 0$$

where $x = (1+i)^{-1}$, it is clear that hat, irrespective of the signs of the a_j there must be n roots, and hence n possible values of i. 37

Hence, the question about the uniqueness of the internal rate of return, really resolves into the method under which there will be one and only one root to the equation (1), or alternatively, one and only one value for i.

4. Methodology

4.1 In this section, before the main theorem, which actually constitutes the method for the uniqueness of the internal rate of retun, we cite some existed results.

Let f(x) be a real polynomial with roots $r_1, r_2, \dots r_n$. Consider the sequence of polynomials

$$f(x) = (x-r_1)(x-r_2)\dots(x-r_n) = x^n + a_1x^{n-1} + \dots + a_n$$
$$f'(x) = \sum (x-r_1)(x-r_2)\dots(x-r_{n-1})$$
$$f_2(x) = \sum z(r_1, r_2)(x-r_3)(x-r_4)\dots(x-r_n)$$
$$f_3(x) = \sum z(r_1, r_2, r_3)(x-r_4)(x-r_5)\dots(x-r_n)$$

....

$$f_n(x) = z(r_1, r_2, \dots r_n)$$

Then, we can prove [3, pg. 93] that the polynomials (3) contitute a Sturm sequence for the real polynomial f(x).

Now, if one defines

$$u_v = \sum r_i^v / (x-r_i) \qquad (4)$$

$$\text{Evidently, } u_0 = f'(x)/f(x) \qquad (5)$$

$$\text{and} \qquad \sum (x-r_i) r_i^v / (x-r_i) = s_v = xu_v - u_{v+1}$$

$$\text{Hence,} \qquad u_{v+1} = xu_v - s_v \qquad (6)$$

and the u_v can be evaluated recursively.

The following theorem, which is an extension of the Sturm theorem, states: [3, pg. 92]

Theorem: Let f(x) be a real polynomial with roots $r_1, r_2, \dots, r_n$, and define u_v by (4). These satisfy the recursion (6) with (5) as the initial condition.

Then, if in the Sturm sequence of determinants

$$1,\ u_0,\quad \delta\begin{matrix} u_0 & u_1 \\ u_1 & u_2 \end{matrix},\ \ldots\ \delta\begin{matrix} u_0 & u_1 \ldots & u_{n-1} \\ u_1 & u_2 \ldots & u_n \\ \ldots & \ldots & \\ \ldots & \ldots & \\ u_{n-1} & u_n \ldots & u_{2n-2} \end{matrix}$$

V_x is the number of variations is sign, and $b > a$, the difference $V_a - V_b$ is equal to the number of distinct real roots lying between a and b.

The proof of this theorem is based on Sturm's Theorem [3, pg. 89] and on the fact that the sequence of polynomials (3) constitute a Sturm sequence for f(x).

4.2 Consider now the polynomial

$$\varphi(x) = \delta \begin{matrix} s_0x\text{-}s_1 & \ldots\ldots\ldots\ldots s_{n-1}\,x\text{-}s_n \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ s_{n-1}\,x\text{-}s_n & s_{2n-2}\,x\text{-}s_{2n-1} \end{matrix} \tag{7}$$

which is a constant multiple of $f(x) = x^n + a_1x^{n-1} + \ldots. + a_n$.

We define $u_v = \sum r_i^v\,(x\text{-}r_i)$ (8)

which satisfy the recursion

$$u_v = s_v\,x\text{-}s_{v+1} \tag{9}$$

Then, we can show that the polynomials resulting from (6) by deleting in sequence the last row and the last column form a Sturm sequence.

Moreover, these polynomials taken in reverse order can be written as

$$1,\quad \delta\begin{matrix} s_0 & 1 \\ s_1 & x \end{matrix},\ \delta\begin{matrix} s_0 & s_1 & 1 \\ s_1 & s_2 & x \\ s_2 & s_3 & x^2 \end{matrix},\ \ldots \tag{10}$$

By means of Newton identities,

$s_1 + a_1 = 0$
$s_2 + a_1s_1 + 2a_2 = 0$
. . . .
$s_{n+1} + a_1s_{n-2} + \ldots + a_{n-2}\,s_1 + (n\text{-}1)\,a_{n-1} = 0$
$s_n + a_1s_{n-1} + \ldots + a_{n-1}\,s_1 + na_n = 0$

the Sturm sequence (10), is written in the following way:

$$1, \quad \delta \begin{vmatrix} 1 & a_1 & 0 \\ 0 & b_0 & 1 \\ b_0 & b_1 & x \end{vmatrix}, \quad \delta \begin{vmatrix} 1 & a_1 & a_2 & a_3 & 0 \\ 0 & 1 & a_1 & a_2 & 0 \\ 0 & 0 & b_0 & b_1 & 1 \\ 0 & b_0 & b_1 & b_2 & x \\ b_0 & b_1 & b_2 & b_3 & x^2 \end{vmatrix} \quad \ldots \qquad (11)$$

where a_1, a_2, a_3, ..., are the coefficients in the real polynomial

$$f(x)= x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \ldots,$$

and b_0, b_1, b_2, ..., (for simplicity of notation $b_j = (n-j)\, a_j$) are the coefficients for the polynomial

$$f'(x) = b_0x^{n-1} + b_1x^{n-2} + b_2x^{n-3} + \ldots$$

The Main Theorem: Let the investment proposal be characterized by the sequence of real numbers a_j ($j= 0, 1, 2, \ldots, n$), where each a_j represents the net cash flow generated by the investment in j-th period of its life.
Let the present value function

$$P(i) = \sum_{j=0}^{n} a_j (1+i)^{-j} = \sum_{j=0}^{n} a_j x^j = f(x)$$

and its first derivative

$$f'(x) = b_j x^{j-1} + b_{j-1} x^{j-2} + \ldots + b_1$$

$$\text{where } b_j = j\, a_j,\ b_{j-1} = (j-1)\, a_{j-1}, \ldots\ b_1=a_1.$$

Let, the Sturm sequence of determinants which terminates at $D_j(x)$, be:

$D_0(x) = 1$

$$D_1(x) = \delta \begin{vmatrix} a_j & a_{j-1} & 0 \\ 0 & b_j & 1 \\ b_j & b_{j-1} & x \end{vmatrix}$$

$$D_2(x) = \delta \begin{vmatrix} a_j & a_{j-1} & a_{j-2} & a_{j-3} & 0 \\ 0 & a_j & a_{j-1} & a_{j-2} & 0 \\ 0 & 0 & b_j & b_{j-1} & 1 \\ 0 & b_j & b_{j-1} & b_{j-2} & x \\ b_j & b_{j-1} & b_{j-2} & b_{j-3} & x^2 \end{vmatrix}$$

$$D_3(x) = \delta \begin{vmatrix} a_j & a_{j-1} & a_{j-2} & a_{j-3} & a_{j-4} & a_{j-5} & 0 \\ 0 & a_j & a_{j-1} & a_{j-2} & a_{j-3} & a_{j-4} & 0 \\ 0 & 0 & a_j & a_{j-1} & a_{j-2} & a_{j-3} & 0 \\ 0 & 0 & 0 & b_j & b_{j-1} & b_{j-2} & 1 \end{vmatrix}$$

$$\begin{matrix} 0 & 0 & b_j & b_{j-1} & b_{j-2} & b_{j-3} & x \\ 0 & b_j & b_{j-1} & b_{j-2} & b_{j-3} & b_{j-4} & x^2 \\ b_j & b_{j-1} & b_{j-2} & b_{j-3} & b_{j-4} & b_{j-5} & x^3 \end{matrix}$$

$D_4(x) =$ - - - -

Let V_1 and V_2 be the number of variations in sign in the sequence $D_0(0), D_1(0), \ldots, D_j(0)$, and $D_0(\infty), \ldots, D_j(\infty)$, respectively. Then, the internal rate of return for the ivestment is unique, if $|V_1 - V_2| = 1$ and the number of multiple rates of return $= |V_1 - V_2|$.

4.3 Example:

The point of this example is to illustrate the method. Assume that an investment proposal has the following cash flows

\$600	\$100	\$125	\$425
1	2	3	

Then, the Present Worth equation takes the form:
$P(i) = 100\ (1+i)^{-3} - 425(1+i)^{-2} + 600(1+i)^{-1} - 125$ or
$f(x) = 100x^3 - 425x^2 + 600x - 125$
Hence, $a_0 = -125$, $a_1 = 600$, $a_2 = -425$, $a_3 = 100$
and $b_1 = 600$, $b_2 = -850$, $b_3 = 300$
Then,
$D_0(x) = 1$

$$D_1(x) = \delta \begin{matrix} 100 & -425 & 0 \\ 0 & 300 & 1 \\ 300 & -850 & x \end{matrix} = 30.000x - 42500$$

$$D_2(x) = \delta \begin{matrix} 100 & -425 & 600 & -125 & 0 \\ 0 & 100 & -425 & 600 & 0 \\ 0 & 0 & 300 & -850 & 1 \\ 0 & 300 & -850 & 600 & x \\ 300 & -850 & 600 & 0 & x^2 \end{matrix}$$

$$= 22\ (10^8)x^2 - 184\ (10^8)x + 67(10^8)$$

$$D_3(x) = \delta \begin{matrix} 100 & -425 & 600 & -125 & 0 & 0 & 0 \\ 0 & 100 & -425 & 600 & -125 & 0 & 0 \\ 0 & 0 & 100 & -425 & 600 & -125 & 0 \\ 0 & 0 & 0 & 300 & -850 & 600 & 1 \end{matrix}$$

0	0	300	−850	600	0	x
0	300	−850	600	0	0	x^2
300	−850	600	0	0	0	x^3

$$= -470(10^{12})x^3 + 4.512(10^{12})\, x^2 - 9.494(10^{12})\, x + 33.016(10^{12})$$

Consequently,

$V_1 = 2$, $V_2=1$ and $V_1 - V_2 = 1$.

Thus, x is unique in the interval $(0, \infty)$, and the rate of return on this investment is unique in the interval $(-1, \infty)$.

5. Conclusion

Economists are acquainted with the objection to and limitations of using internal rates of return as a selection criterion among investments projects.

It is now well known that an income-earning investment may have multiple internal rates of return, if the Present Worth equation has more than one reversals of sign [2].

In the Investment literature, answers to this problems are usually given by the Descartes's rule of signs.

It is to emphasize at this point that the Descrates's rule, still leaves an uncertainty as to the exact number of real roots in the Present Worth equation; it only gives an upper limit to them.

The discussed method, not only gives the uniqueness of the Internal rate of return but also is applicable for n-th degree equation, while algebraic methods are not always available for $n \geqslant 5$. In addition this method is extremely suitable for calculations on the computer.

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