ORDINARY, INDIRECT, TWO-STAGE, THREE-STAGE LEAST SQUARES AND INSTRUMENTAL-VARIABLE ESTIMATORS — A UNIFIED EXPOSITION

By

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Introduction

It is, by now, traditional in Econometrics textbooks (see, for example, Johnston (1972), Theil (1971) and Drettakis (1975a, 1979) to cover first Ordinary Least Squares (OLS) and Aitken Generalised Least Squares (GLS) and only then consider Indirect Least Squares (ILS), Instrumental Variables (IV), Two-Stage Least Squares (2-SLS) and finally Three-Stage Least Squares (3SLS) in the context of a Cowles Commission Model.

In this paper a unified exposition is attempted for the derivation of the OLS, ILS, IV, 2SLS and 3SLS estimators by applying the method which has been used by the author (1973) in an earlier note. Rather than adopting the notation of the wellknown textbooks we use here the original notation employed by Koopmans, Rubin and Leipnic (1950) and Koopmans (1953).

I. The Model

The model can be written in the form:

$$y = X\alpha + u \qquad (1)$$

where

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* This exposition owes a great deal to J.D. Sargan's teaching at the London School of Economics and political Science and has some afinities with the exposition used by Dhrymes (1970). I would like to thank D.F. Hendry for his comments on an earlier version of this paper.

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} ; X = \begin{bmatrix} X_1 & & & \\ & X_2 & & 0 \\ & & \cdot & \\ & O & \cdot & \\ & & & \cdot \\ & & & X_n \end{bmatrix} ; \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} ; u = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \quad (2)$$

and where

y_i is a Tx1 vector of the ith endogenous variable, u_i a Tx1 vector of the disturbances of the ith equation, $X_i = (Y_i : Z_i)$ a $Tx(n_i + m_i)$ matrix of the n_i endogenous (Y_i) and m_i predetermined (Z_i) variables included in the ith equation (identification conditons restrict the values of the n_i and m_i) and $\alpha_i' = (\beta_i', \gamma_i')$ a $1x(n_i + m_i)$ vector of the corresponding coefficients ($i = 1, 2, \ldots\ldots, n$). (3)

The following assumptions are made about the disturbances of the model defined by (1), (2) and (3)

$$E(u_i) = o, \; E(u_i u_j') = \sigma_{ij} I_T \text{ and } E(uu') = \Sigma \otimes I_T \quad (4)$$

Where the subscript on the identity r.atrix denotes its order, $\Sigma = \{\sigma_{ij}\}$ is an nxn positive definite matrix, assumed knov'n and $\otimes$ is the Kronecker Product symbol (see Drettakis (1975)).

It is also assumed that th^e probability limits (or the ordinary limits, as the case may be) of the matrices

$$T^{-1} Z'Z, \; T^{-1} Z'Z_i \text{ and } T^{-1} Z'Y_i \quad (5)$$

exist.

Finally in Sections II and III below Z, a Txm (m<T) matrix of all the predetermined variables in the system, with full column rank, is assumed "fixed". (6)

II. Estimation of one equation

The equation to be estimated is:

$$Y_i = X_i \alpha_i + u_i = Y_i \beta_i + Z_i \gamma_i + u_i \quad (7)$$

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CASE II.1: $\beta_i = o, \; \gamma_i \neq o$

In this case (7) becomes:

$$Y_i = Z_i \gamma_i + u_i \quad (8)$$

Using a notation similar to that employed by Sargan (1964) let:

$$\gamma_i^{*\prime} = (1, -\gamma_i'), \; Z_i^* = (y_i \; : \; Z_i) \tag{9}$$

Substituting (9) into (8) we obtain:

$$Z_i^* \gamma_i^* = u_i \tag{10}$$

Given (4) and (6):

$$E(Z_i' u_i) = o \tag{11}$$

Given (4) and (11) the OLS procedure consists of minimising:

$$f_i^* = \gamma_i^{*\prime} Z_i^{*\prime} Z_i^* \gamma_i^* \tag{12}$$

as a function of the unknown parameters γ_i.

Setting the partial derivatives of (12) with respect to γ_i equal to zero and rearranging we obtain the OLS estimating equations:

$$Z_i' Z_i c_i = Z_i' y_i \tag{13}$$

where c_i the unbiased estimate of γ_i.

CASE II.2: $\beta_i \neq o$, $\gamma_i \neq o$, equation (7) overidentified

In this case either IV or 2SLS can be used. As, asymptotically, the estimates obtained by these two methods are identical they will be considered together here. Letting:

$$(Z' Z)^{-1} = KK' \text{ and } M' = K'Z' \tag{14}$$

Premultiplying (7) by M' we obtain:

$$M' y_i = M' X_i \alpha_i + M' u_i \tag{15}$$

By (4), (5), (6) and (14):

$$\begin{aligned} E(M' u_i) = o, \; E(M' u_i u_i' M) = \sigma_{ii} I_T, \\ \text{plim}(^{-1} Z_i' MM' u_i) = o \end{aligned} \tag{16}$$

Proceeding as in the CASE II.1 let:

$$\alpha_i^{*\prime} = (1, -\alpha_i'), \; X^* = M'(y_i \; : \; X_i), \; u_i^* = M' u_i \tag{17}$$

Using (17), equation (15) can be written as:

$$X_i^* \alpha_i^* = u_i^* \tag{18}$$

Given (16) we proceed as in CASE II.1 and minimise:

$$g_i^* = \alpha_i^{*\prime} X_i^{*\prime} X_i^* \alpha_i^* \tag{19}$$

as a function of the unknown parameters α_i. Setting the partial derivatives of (19) with respect to α_i equal to zero and re-arranging we obtain the IV-2SLS estimating equations:

$$X_i' Z(Z'Z)^{-1} Z' X_i a_i = X_i' Z(Z'Z)^{-1} Z' y_i \tag{20}$$

wherea a_i is the consistent estimate of α_i.

CASE II.3: $\beta_i \neq o$, $\gamma_i \neq o$ equation (7) just identified

In this case (20) simplifies to:

$$Z' X_i a_i = Z'_i \tag{21}$$

i.e. the ILS estimating equations in the case where the square matrix $X_i' Z$ is non-singular.

III. Estimation of the system as a whole

Here, again, three cases can be distinguished

CASE III.1: $\beta_i = o$, $\gamma_i \neq o$, all i

In this case (1) can be written as:

$$y = Z^+ \gamma + u \tag{22}$$

where

$$Z^+ = \begin{bmatrix} Z_1 & & & O \\ & Z_2 & & \\ & & \ddots & \\ O & & & Z_n \end{bmatrix} \qquad \gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \cdot \\ \cdot \\ \cdot \\ \gamma_n \end{bmatrix} \tag{23}$$

This, of course, is the case of seemingly unrelated regressions.

Given (4) Σ^{-1} can be written as:

$$\Sigma^{-1} = P'P \tag{24}$$

where P is an nxn non-singular matrix.

Premultiplying (22) by:

$$N' = P \otimes I_T \tag{25}$$

and given (4) and (6) we proceed as in the CASE II.1 and obtain the estimating equations:

$$Z^{+\prime} (\Sigma^{-1} \otimes I_T) Z^{+} c = Z^{+\prime} (\Sigma^{-1} \otimes I_T) y \tag{26}$$

where c is the unbiased estimate of γ.

CASE III.2: $\beta_i \neq o$, $\gamma_i \neq o$, all i and some equations overidentified

Premultiplying (1) by:

$$M^{*\prime} = P \otimes M' \tag{27}$$

and given (4), (5), (6) and (24) we proceed as in the CASE II.2 and obtain the IV-3SLS estimating equations:

$$X' (Z^{**} (\Sigma^{-1} \otimes (Z'Z)^{-1}) Z^{**\prime} Xa = X' Z^{**} (\Sigma^{-1} \otimes (Z'Z)^{-1}) Z^{**\prime} y \tag{28}$$

where:

$$Z^{**} = I_n \otimes Z \tag{29}$$

and a is the consistent estimate of α.

CASE III.3: $\beta_i \neq o$, $\gamma_i \neq o$, all i and all equations just identified

In this case (28) simplifies to:

$$Z^{**\prime} Xa = Z^{**\prime} y \tag{30}$$

which are the ILS estimating equations for the full system in the case where all the square matrices:

$$X_i' Z, \; i = 1, 2, \ldots\ldots\ldots, n$$

are non-singular.

IV. Lagged endogenous variables in the Z matrix

The application of the procedure outlined in Sections II and III in the case examined in this Section requires the use of the result (see Theil (1971), pp. 484-88) that:

$$T^{-(1/2)} Z^{**\prime} u \tag{31}$$

has a normal limiting distribution with zero mean vector and covariance matrix:

$$\Sigma \otimes \text{plim}(T^{-1} Z'Z) \tag{32}$$

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