A MATHEMATICAL PROOF OF THE EQUIVALENCIES OF THE SINKING FUND AND RETURN ON INVESTED CAPITAL METHODS

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Introduction

The multiple-rate of return problem was one of the most widely debated issues in the field of capital investment analysis during the late 1950's and early 1960's. The recent popularity of leveraged leasing with its "non-conventional" investment aspects has generated renewed interest in this topic.

Two of the most popular methods of determining a single meaningful rate of return index in non-conventional investment problems are the sinking fund method and the return on invested capital (RIC) method.

Despite their apparent differences, we demonstrate that these two methods will always lead to identical numerical results when applied to unconventional investments having two reversals of signs. Only when there are three or more reversals of signs will the methods result in different numerical return indexes.

I. Unconventional Investments with Two Reversals of Signs

Let's assume a non-conventional investment of the general form:

$$-a_0,\ +a_1,\ +\ldots,\ +a_{h-1},\ -a_h,\ -\ldots,\ -a_n \tag{1}$$

where all a_i are positive, $i=1, \ldots, n$, and h is the first negative term following the inflows.

Applying the RIC method we have the following [6]:

Step 1: Find r_{min} by the trial-and-error method. The r_{min} is defined in such a way that the project balances $S_t(r_{min})$ are all zero or negative for $t = 0, 1, \ldots, n-1$.

Step 2: Find $S_n(r_{min})$.

a. If $S_n(r_{min}) \geqslant 0$, then the project is a pure investment.
   (1) Find IRR, r^*, such that $S_n(r^*) = 0$
   (2) Algorithm is complete.

b. If $S_n(r_{min}) < 0$, then the project is a mixed investment so continue with Step 3.

Step 3: Let the cost of capital equal k.

Step 4: Calculate $S_t(r, k)$ according to the rule

$$S_0(r, k) = a_0$$

$$S_1(r, k) = S_0(1+r) + a_1 \quad \text{if } S_0(r, k) < 0 \tag{2}$$

$$= S_0(1+k) + a_1 \quad \text{if } S_0(r, k) > 0 \tag{3}$$

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$$S_n(r, k) = S_{n-1}(1+r) + a_n \text{ if } S_{n-1}(r, k) < 0 \tag{4}$$

$$= S_{n-1}(1+k) + a_n \text{ if } S_{n-1}(r, k) > 0 \tag{5}$$

Step 5: Determine the value of r by solving the equation $S_n(r, k) = 0$.

In (1) the first balance, $S_0 = -a_0$ is negative. S_1 is derived by multiplying S_0 by the positive number $(1+r)$ and adding the positive number a_1.

Hence, $S_1 = S_0(1+r) + a_1$ which is either positive or negative depending on the magnitudes of a_0 and a_1. Let S_f be the first non-negative balance we compute, i.e., $S_f \geqslant 0$ and $S_{f-1} < 0$.

Theorem 1: The term a_f is one of the positive terms.

Proof. We know that $S_{f-1} < 0$

$$\text{or } S_{f-1}(1+r) < 0 \tag{6}$$

$$\text{or } -S_{f-1}(1+r) > 0$$

$$\text{also } S_f = S_{f-1}(1+r) + a_f > 0 \tag{7}$$

by adding (6) + (7): $-S_{f-1}(1+r) + S_{f-1}(1+r) + a_f > 0$ or $a_f > 0$.

Theorem 2: All balances between S_f and S_n will be non-negative.

Proof. S_{f+1} is positive because it is derived by multiplying S_f by the positive number $1+k$ and adding the positive number a_{f+1}

$$S_{f+1} = S_f(1+k) + a_{f+1} > 0 \tag{8}$$

For the same reason the balances between S_f and S_{h-1} are all positive.

Now let us proceed by proving that the balances $S_h, S_{h+1}, \ldots, S_{n-1}$ are non-negative also. We will assume for a moment that there is a negative balance S_m where $h < m < n$.

S_{m+1} is derived by multiplying S_m by the positive number $(1+r)$ and adding the negative number $-a_{m+1}$. So, S_{m+1} is negative also.

$$S_{m+1} = S_m(1+r) - a_{m+1}$$

S_{m+2} is negative for the same reason and so will be the remaining balances including

S_n, since all a_i's for $m<i<n$ are negative. However, this conclusion contradicts the fact that $S_a(r, k) = 0$. Therefore the S_i's for $h \leqslant i<n$ cannot be negative.

Theorem 3: The series of project balances, S_i, changes signs only once.

Proof. The first balance, $S_0 = -a_0$ is negative. The balance S_f is the first positive balance in the series. As it was shown in theorem 2, all balances between S_f and S_n are positive. Therefore, the only change of signs (from negative to positive) occurs at S_f.

To evaluate the polynomial (1) by the Sinking Fund Method the following steps should be taken:

Step 1: We determine the series of cash flows:

$$a_d, \; +a_{d+1}, \; +\ldots \; -a_h, \; -\ldots, \; -a_n$$

so that when discounted at the cost of capital its present value at year d (where the series initiates) is greater than or equal to zero.

$$P_d = a_d + \frac{a_{d+1}}{(1+k)} + \ldots - \frac{a_h}{(1+k)^{h-d}} - \ldots - \frac{a_n}{(1+k)^{n-d}} \geqslant 0 \tag{9}$$

The significance of equation (9) is that all P_i's for $d<i \leqslant n$ are negative. Hence:

$$P_{d+1} = a_{d+1} + \frac{a_{d+2}}{(1+k)} + \ldots - \frac{a_n}{(1+k)^{[(n-d)-1]}} \leqslant 0 \tag{9'}$$

Step 2: The Sinking Fund rate of return, r', is calculated by solving the remaining portion of the original series of cash flows conventionally, using the Internal Rate of Return. This sub-series is:

$$-a_0 + \frac{a_1}{(1+r')} + \frac{a_2}{(1+r')^2} + \cdots + \frac{P_d}{(1+r')^d} = 0 \tag{9''}$$

So far we have evaluated polynomial (1) by the RIC and SF methods. We have solved for the rate of return as a function of the Cost of Capital. Now we pose the following question: In a cash flow similar to the one represented by polynomial (1), are the yields r and r' of the RIC and SF methods respectively, the same? The answer is yes, and it will be proven as follows.

In the RIC method, according to the theorems (1) & (2) there is a year f, $0<f<h$ where the project balances become non-negative for the first time and they remain non-negative until the last balance, $S_n = 0$.

Hence, $S_f>0$ and $S_{f-1}<0$ or

$$S_f = -a_0 (1+r)^f + a_1 (1+r)^{f-1} + \ldots + a_f > 0 \tag{10}$$

and

$$S_{f-1} = -a_0 (1+r)^{(f-1)} + a_1 (1+r)^{f-2} + \ldots + a_{f-1} < 0 \tag{11}$$

The last balance is:

$$S_n = -a_0(1+r)^f(1+k)^{(n-f)} + a_1(1+r)^{f-1}(1+k)^{(n-f)} + \qquad$$
$$+ \ldots + a_f(1+k)^{n-f} + a_{f+1}(1+k)^{[(n-f)-1]} + \ldots - \ldots - a_n = 0 \qquad (12)$$

Multiplying (10) by the positive number $(1+k)^{(n-f)}$ we have:

$$S_f(1+k)^{(n-f)} = S_f' = -a_0(1+r)^f(1+k)^{n-f} + a_1(1+r)^{f-1}(1+k)^{n-f}$$
$$+ \ldots + a_f(1+k)^{n-f} > 0 \qquad (13)$$

Subtracting (13) from (12) we have:

$$S_n - S_f' = a_{f+1}(1+k)^{[(n-f)-1]} + \ldots - \ldots - a_n < 0$$

and dividing the resulting equation by $(1+k)^{[(n-f)-1]}$ we get:

$$S_{f+1}' = a_{f+1} + \frac{a_{f+2}}{1+k} + \ldots - \ldots - \frac{a_n}{(1+k)^{[(n-f)-1]}} < 0 \qquad (14)$$

By multiplying inequality (11) by $(1+r)(1+k)^{n-f}$, subtracting the resulting equation from (12) and dividing by $(1+k)^{(n-f)}$ we find:

$$S''_f = a_f + \frac{a_{f+1}}{1+k} + \ldots - \ldots - \frac{a_n}{(1+k)^{(n-f)}} > 0 \qquad (15)$$

By comparing and contrasting inequalities (14) and (15) with (9′) and (9) respectively and because of theorems 1, 2 and 3, we conclude that $P_d = S''_f$ and $P_{d+1} = S''_{f+1}$ or d=f for the same k.

In the Sinking Fund method we use equation (9″) to solve for r′ as a function of the Cost of Capital.

$$-a_0 + \frac{a_1}{(1+r')} + \ldots + \frac{P_d}{(1+r')^d} = 0 \qquad (9'')$$

or

$$-a_0 + \frac{a_1}{(1+r')} + \ldots + \frac{a_f + \cdots - \cdots - \frac{a_n}{(1+k)^{n-f}}}{(1+r')^f} = 0$$

Multiplying both sides by $(1+r')^f(1+k)^{n-f}$ we have:

$$-a_0(1+r')^f(1+k)^{(n-f)} + a_1(1+r')^{f-1}(1+k)^{(n-f)} + \ldots +$$
$$a_f(1+k)^{n-f} + a_{f+1}(1+k)^{[(n-f)-1]} + \ldots - \ldots - a_n = 0 \qquad (16)$$

Comparing polynomials (12) and (16) we conclude that $r=r'$ for every value of k and a_i. Hence, the RIC and SF methods when applied to cash flows of the type (1) for every value of k will yield consistently the same rates.

II. UNCONVENTIONAL INVESTMENTS WITH THREE OR MORE REVERSALS OF SIGNS

Let us assume a non-conventional project with three reversals of signs of the general form:

$$-a_0,\ +a_1,\ +a_2,\ +a_3,\ \ldots,\ -a_h,\ \ldots,\ -a_{n-1},\ +a_n$$

The RIC Algorithm, all five steps, will be used in this type of problem the way it was used in the case of two reversals of signs. Whenever S_t is negative it will be compounded by the rate of return and whenever positive by the cost of capital.

Since the last cash flow, a_n, is positive, that means that balance S_{n-1} is always negative and therefore, will be compounded by $(1+r)$, added to a_n, and set equal to zero, before the rate of r is solved as a function of k.

$$S_{n-1}(1+r) + a_n = 0 \tag{17}$$

The final equation as derived by the RIC method will differ from the one derived by the Sinking Fund method. The negative cash flows, $-a_h, \ldots, -a_{n-1}$, in the sinking fund method will be discounted by the sinking fund rate until "paid-off" and no "portion" of these negative outflows will be thought as the investment that will generate a_n.

This will result in the following type of polynomial:

$$-a_0 + \frac{a_1}{(1+r')} + \frac{a_j}{(1+r')^j} + \frac{a_n}{(1+r')^n} = 0 \tag{18}$$

In conclusion, a comparison of equations (17) and (18) reveals that the RIC and Sinking Fund methods when applied to this type investments will not yield the same rates of return, except in the rare instance when the rate of return is equal to the cost of capital/sinking fund rate. The reason for the different results is attributed to the fact that the investment and financing portions of the project are defined differently by each method. It is of interest to note, however, that experimentation with the two methods has indicated that the difference in numerical results are generally not significant.

Conclusions

This paper has demonstrated that the RIC and sinking fund methods will always lead to identical numerical results when applied to unconventional cash flows having two reversals of signs. When the cash flows involve three or more sign changes, the two methods will generally not yield the same rates of return. Although our experience with a wide variety of problems indicates that both methods will always make the same accept or reject decision, further research is warranted in this area.

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