POINT PROCESSES AND THE SIZE DISTRIBUTION OF ECONOMIC VARIABLES

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DEFINITION OF A STOCHASTIC PROCESS

Cox and Miller[1] define a stochastic process as a system which develops in time or space according to probabilistic laws, time being the most common parameter of the process. Massy et al[2] emphasize the ex ante probabilistic structure of the stochastic models, we quote : «As we understand current terminology, a stochastic model is a model in which the probability components are built in at the outset rather than being added ex post facto to accommodate discerpancies between predicted and actual results. That is, the probabilistic components form an important part of the basic structure of the stochastic model».

There have been a lot of criticisms related to the application of stochastic models into economics. The main criticism has been directed at the economic content of the models. Many economists.* argue that the element of chance has become a substitute for economic theory, thus questioning or even rejecting the relevance of stochastic models as explanations of size distributions. Although the random elements are inherent in both the conditions and the assumptions under which the economic system has to function the stochastic models so far presented incorporate few of the familiar economic variables, and consequently these models set up to explain size distributions lack in economic content ; we quote from Shorrocks[5] «to some extent the problem is one of defining suitable

* See, for instance, LydalP and Sahota[4]

stochastic analogues of deterministic theories, a problem which is aggravated when state variables, such as wealth classes, do not correspond to those used in deterministic models».

Let us try to clarify a basic misunderstanding that prevails among economists as far as the idea of randomness is concerned. It is a well - known fact in the literature that mathematical models are either deterministic or probabilistic (stochastic models falling under the latter category). In deterministic models we consider mathematical variables and the effect of any change in the system can be predicted with certainty. On the other hand, whenever the system is not fully specified or because of the unpredictable character of the human behaviour (and this is the case in economics) there is a certain uncertainty incorporated in any prediction or outcome, and so in this case we introduce random variables with probability distributions assigned to them. The introduction of random variables though does not necessarily mean that mere chance plays the important role in the system as many economists still believe. Mere chance or randomness, being the proper term for it, operates in a particular class of stochastic processes and it is directly related to the Markovian property and the lack of memory of the geometric and the negative exponential distributions, as we will explain later.

Bartholomew[6] mentions a more general objection that it is often raised to the application of stochastic models in the social sciences. The argument that by allowing probability laws to govern human relations we deprive human beings of freedom of choice is not theoretically valid and, according to the author, it rests on a misunderstanding of the nature of probability theory in model building ; since human beings are free-decision markers, their bahaviour is unpredictable and, therefore a probabilistic model is more appropriate to be applied, while a deterministic one would constrain human behaviour along a predetermined path. Bartholomew also markes a distinction between «explananory» models and «black - box» ones. He defines the former as the ones that aim to explain the mechanism which governs the relationship between the input and the output variables of a certain system, while the latter merely state the relationship, a model of the regression type being a typical example.

II. THE HOMOGENEITY ASSUMPTION AND THE MARKOVIAN PROPERTY

Stochastic processes are basically divided according to two criteria, the assumption of homogeneity and the Markovian property, which have been the main obstacles to the introduction of economic theory into the stochastic models so far presented. Suppose we have a Markov process with discrete states in continuous time and let X (t) demote the state occupied at time t. The process is in priciple defined when we have a set of functions (transition probability functions) Pij(t) with the interpretation :

Pij(t) = probab {X(t+u) = j | X(u) = i} and the process is time - homogeneous if the probability functions are independent of u. The homogeneity assumption has enabled economists to make extensive use of the properties of ergodic processes and equilibrium aspects of the systems ; but if the parameters of the system are functions of economic variables, generally time - dependent, the time - homogeneous models cannot by nature explain the size distributions of income or wealth.

Shorrocks[7]. criticized the time-homogeneous models analytically and he introduced a non - homogeneous birth and death process with «immigration» in order to explain the size distribution of wealth, the transition intensities of the process being dependent upon the individual's age t.

Although representing the process parameters as functions of time - dependeni economic variables seems to be a formidable task, it does not give a satisfactory answer to the objections raised so far; as Quandt[8] in his model of the size distribution of firms points out, we quote, «what guarantee or what reason for belief does one have that the transition matrix in one industry will be the same, or nearly the same as in another? In general one cannot assume the inter-iuduatry stability of transition matrices. If one is willing to hypothesize, contrary to what some have held, that cost and demand functions have something to do with the manner in which industries develop, one may arrive at a model of industry development in which transition matrices depend on the following factors :

1. The nature of the short- run cost function. . . . . .
2. The nature of the long-run cost function. . . . . .
3. The nature of oligopolistic arrangements - or the absence thereof-in a given industry. . . . . .
4. The general configuration of competing products, changes in relative teschnology, and changes in relative demands. . . . . .

But it is not even plausible to suggest that only the values of certain parameters will be different ; in all likelihood the nature of the stochastic process itself will differ from case to case. Accordingly it would be surprisingg if the same distribution (with different estimated values for the parameters) were to fit all cases».

Ruggles[9] emphasized the fact that a wide range of institutional, historical and sociological factors are directly related to the overall distribution of incomes, while Shorrocks[10] says the following, «the Markov property implies that individuals with the same income will have identical predictions for the future, regardless of whether in the recent past their income has been increasing or falling, and it denies the possibility of serial correlation in the growth rates for different periods. This is incompatible with Friedman's distiction between permanent and transitory income which predicts a negative correlation between growth rates of income». Consequently, so far as the application of stochastic processes into economics is concerned it seems that the Markovian property plays an important role.

The Markovian property can be described as follows : suppose we have a discrete time stochastic process $\{X(t), t= 0,1,2,.. ,\}$ or a continuous parameter one $\{X(t), t \quad 0\}$, we call the process Markovian if, for any set of *n*, time points $t_1 < t_2 < . . . < t_n$, the conditional distribution of X(tn), for given values of (t_1), $X(t_2),. . .X(t_{n-1})$, depends on $X(t_{n-1})$, the most recent known value. In other words prob. $\{X(t_u) \quad x_n \mid X(t_1) \quad x_1, \; ... \; X(n-1)=xn-1\}=$

$=$prob. $\{X(t_n) \quad x_n \mid X(t_{n-1}) = X_{n-1}\}$ and so the present of the process determines its future, the past being of no importance. However, we can examine the property of the same process not by studying the random variable X(t) but by considering the points of time where transitions occur ; in order to examine the latter approach we have to introduce a particular class of stochastic processes, the point processes.

III. POINT PROCESSES AND THEIR APPLICATIONS

Point or counting * processes are a general type of non - Markovian process. In a point process there are certain events which possess the regenerative quality, that is once a regenerative point has been reached, the whole development preceding and leading up to this point (or event) is of no further consequence for the following process. The regenerative event embodies the whole past, that is any information relevant for the further development of the process.More precisely, suppose that a process $\{X(t), t \quad 0\}$ is such that for some particular time T and for all $t > T$ the conditional probability distribution of X(t), given X(T), is equal to the conditional probability distribution of $X(\tau)$, given X(t), for all $\tau \quad T$, the point T or the event by which it can be identified is called a regeneration point for the process and any process that has such points is called a counting or point process.

* As Parzen[11] refers to them.

Markov processes are those in which every point along the time-axis has the regenerative quality, thus a point process being a generalization of the Markov one.

A very important category of point processes in the well - known renewal process; an integer - valued process $\{n(t), t>0\}$· corresponding to a series of points or events distributed on the interval (0,t) is called a renewal process if the inter-arrival times $\tau_1, \tau_2,\ldots$ between successive events are independent identically distributed positive random variables. The proper random variable n(t) is the number of renewals or events during the interval (0,t), and it has finite moments of all orders.

Let Sn be the waiting time to the nth event, that is the time it takes for η events to occur. The successive inter-arrival times are defined as follows:

τ_1 is the time from zero to the first event and for $i>1$,
τ_i is the time from the (i - l)st to the ith event.

Therefore $S_n = \tau_1+\tau_2 4-\ldots\tau_n$ for n 1. There is a basic relation between a point process {n(t), t 0) and the corresponding sequence of waiting times $\{S_n\}$. For $t>0$ $n(t)=m.ax \{n|S_n \; t\}$, or $n(t)<n$ if and only if $S_{n+1}>t$, n= 1,2,. . . It follows that $n(t)=n$ if and only if $S_n<t$ and $S_{n+1}>t$.

The Markovian property is uniquely related to a particular revewal process, the Poisson process. Let η (t, t+Δt) denote the number of events in the interval (t, t+Δt). Suppose that, for some positive constant p. as Δ t 0 , we have :

$$\text{prob. } \{n(t, t+ At) = 0\} = 1 - p(\Delta t)+0(\Delta t), \quad (1)$$

the last term implying that the probability of more than one events occuring simultaneously is zero in the interval Δt. Also.

$$\text{prob. } \{n(t, t+ \Delta t) =1\} == p(\Delta t) + 0(\Delta t), \quad (2)$$

therefore n(t, t + Δt) is completely independent of occurrences in the interval (0,t]. We call this stochastic series of events a Poisson process of rate p. Let us now consider a new time origin at t_0, which may be a point at which an event has just occurred, or any fixed point. Let t_0+z be the time of the first event after t_0, and let us calculate the probability distribution of the random variable z. If P(x) = prob. (z>x) it follows that P (χ+Δχ) = prob. (z>x) times the

$$\text{prob. } \{(\text{no event occuring in } t_0+ \chi, (t_0+\chi+\Delta\chi) \mid z>x\backslash \quad (3)$$

Equation (3) holds for any renewal process but a special feature of the Poisson process that describes the Markovian property is that the conditional probability in (3) is not affected by the condition z> x, which refers to what happens at or before t_0+x. Therefore all the properties of the Poisson process referring to its future behaviour, after t_0, are independent of what happens at or before t_0. In this case we can replace the conditional probability in (3) with the unconditional one in (1) so that (3) becomes :

$$P(\chi+\Delta\chi) = P(\chi)\ \}(1 - \rho\ (\Delta\chi)+0(\Delta\chi)\}$$

or

$$P'(\chi) = - \rho P(\chi) \quad (4)$$

The solution of equation (4) is $P(\chi) = P(0)\ e^{-px}$, with the initial condition P(0) — prob. (z>0)=1. Finally we obtain $P(x) == e^{-px}$, and the probability density function of ζ is $pe{\sim}^{xp}$, x>0, the is the negative exponential density with parameter p. Since χ is a random variable corresponding to the time between one event and the next one, the probability that the waiting time until the next event is longer than χ is e^{-px}, the exponential law for the waiting time being the only continuous distribution said to be endowed with complete «lack of memory»[12]; in other words, the pobability that we have to wait χ time units for the next event is independent of the time we have already waited since the last event occurred. In the discrete case the role played by the exponential distribution is assumed by the geometric one.

Summing up, the Poisson process is fully determined by the negative exponential law and events occurring in the process are referred to as events occurring at random. While in statistics the word «random» indicates independence of observations, the term here denotes that the Poisson process assumes independence of occurrence of events so that an infinite number of points (events) is randomly distributed over the interval [0,). Any Markov process, as we noted earlier on, is characterized by the exponential (or the geometric) law in the sense that the underlying point process, the process counting the number of transitions, is of the Poisson type.

Point, or more specifically, renewal processes have been increasingly applied to the various branches of the social sciences. Thus, Coleman" emphasized the importance of the Poisson process in describing various problems in sociology, and Bartholomew[14] thoroughly examined the application of renewal processes to the turnover of people in an organization; finally, Massy et al[15] introduced a non - homogeneous Poisson model where purchasing decisions made by individual families are represented as discrete events occurring across the time axis. Nevertheless, the role of point processes in economics has never been examined before. Only Steindl[16] tried to draw attention to the role of these processes, we quote, «We have obvious regeneration points in the pay-daye when wages and salaries are received, or in the accounting periods of firms. Moreouer, all consumption is a renewal process, as also is production».

Although one can hardly believe that economic phenomena can be necessarily described by a Markov process, the known models representing stochastic processes that generate distributions of economic variables have the Markov property. This may be due to the existing elegant mathematical theorems of the Markov processes, while point processes are difficult to handle. The relevance of the negative exponential law in the underlying point processes has never been questioned. It may be true that this assumption reasonably describes actual phenomena such as the length of the telephone conversations within a city, or the duration of machine repairs. But when it comes to size distribution of economic variables, such as income or earnings, it is not at all certain whether the «lack of memory» assumption is theoretically and empirically supported or not, this particular area has been comletely unexplored. For instance, with respect to the distribution of earnings of an individual in a large bureaucratic corporation a plausible model cannot be merely represented or generated by the Poisson process. If we only apply this process then the employee's earnings are likely to change at every point of his lenght- of- service ; that is the probability of the time elapsing until the next change of his earnings is independent of the time already elapsed since the last change occurred. Obviously, only at certain points of the individual's length-of- service a change of his earnings takes place and this is related to the wage structure which the individual is subject to.

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