DERIVING THE NORMAL DENSITY AS A SOLUTION OF A DIFFERENTIAL EQUATION

VASSILIOS C. HOMBAS
Univercity of Athens

This paper shows how we can produce the normal density using a differential equation. The key idea is that the subtangent to every point of the curve, be an appropriate expression of the random variable.

Key Words : Normal density ; Standard Normal ; Differential equation.

1. INTRODUCTION

The normal distribution was studied initially by A. De Moivre (1667- 1745) and P. Laplace (1749- 1827). Gauss in his «Theoria Motus Corporum Coelestium» suggested that the distribution of errors, supposed to be continuous, is normal (see, Whittaner and Robinson, 1926, § 112).

Although the normal density is the cornerstone in statistics, we have not yet seen a derivation of the density from a differential equation by a geometric inspection. Griffin, Smith and Watts (1982, p. 373 - 376) describe a method of deriving the normal density using EDA techniques.

During my early years in studying statistics and later on when lecturing on the subject, I found the procedure of teaching the normal density as suggested by most text books cumbersome and somehow anbritarily. The purpose of this paper, therefore, is to show the derivation of the normal density function from a differential equation, which could be used in teaching, i.e. find the equation of the curve whose length of the subtangent at every point is reciprocal to the linear expression $\gamma\chi+\delta$> where χ is the abscissa of the corresponding point of the curve and γ, and δ, are population parameters ($\gamma>0$).

2. DERIVING THE NORMAL DENSITY FUNCTION

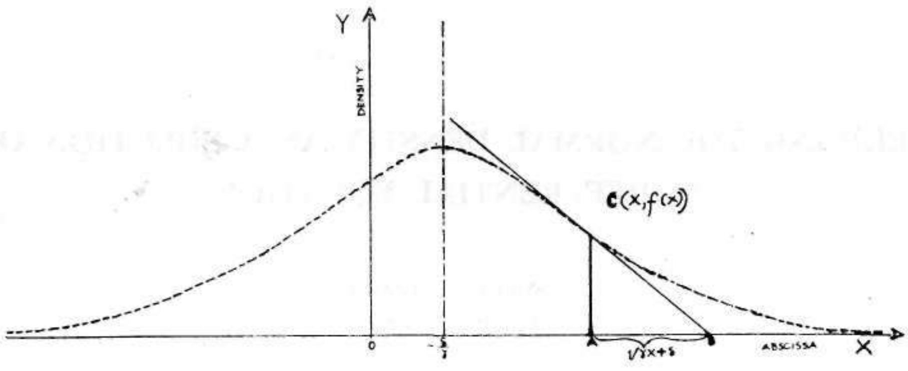

Figure 1. Derivation of the Normal Curve

Let $y = f(x)$ be the equation of the unkown curve. To derive the density function the students are instructed to examine Figure 1 ; and to find a curve so that the length AB of subtangent, is the reciprocal of the linear expression $\gamma\chi+\delta$, where χ is the abscissa of a point of the curve and γ, δ parameters (γ positive).

Looking at the right triangle ABC, the gradient f (x) of the tangent line must be equal to $-(AC)/(AB)$. Thus $(AC) = f(x)$ and $(AB)= 1/(\gamma\chi+\delta)$, therefore the differentiate equation is

$$f'(x) = -(\gamma x+\delta)f(\chi) \quad (1)$$

In order to solve (1), we can use the method of separation of the variables as follows :

$$\frac{dy}{y} = -(\gamma x+\delta)\,dx \quad (2)$$

Integrating both sides

$$\ln(y.c^{-1}) = \delta\chi - \gamma\chi^2/2 \qquad (3)$$

and finally

$$y = C \exp(-\delta\chi - \gamma\chi 2/2) \qquad (4)$$

where c is the constant of integration.

Now, we want to scale the resulting curve (4), to have unit area. So c must be determined so that

$$\int_{-\infty}^{+\infty} c \exp(-\delta x - \gamma x^2/2)dx = 1 \qquad (5)$$

or

$$c \int_{-\infty}^{+\infty} \exp(-\delta x - \gamma x^2/2)dx = 1 \qquad (6)$$

Taking the left side of equation (6) we manipulate as follows.

$$c \int_{-\infty}^{+\infty} \exp(-\delta x - \gamma x^2/2)dx = c \int_{-\infty}^{+\infty} \exp((-\gamma/2)(x+\delta/\gamma)^2) \exp(\delta^2/2\gamma)\, dx$$

$$= c \exp(\delta^2/2\gamma) \int_{-\infty}^{+\infty} \exp((-\gamma/2)(x+\delta/\gamma)^2)\, dx$$

$$= (c/\sqrt{\gamma}) \exp(\delta^2/2\gamma) \int_{-\infty}^{+\infty} \exp((-1/2)(\sqrt{\gamma}x+\beta\sqrt{\gamma}/\gamma)^2)\ d(\sqrt{\gamma}x+\beta\sqrt{\gamma}/\gamma)$$

$$= (c\sqrt{2\pi}/\sqrt{\gamma})\ \exp(\delta^2/2\gamma)$$

Note that $$\int_{-\infty}^{+\infty} \exp((-1/2)(\sqrt{\gamma}x+\beta\sqrt{\gamma}/\gamma)^2)\, d(\sqrt{\gamma}x+\beta\sqrt{\gamma}/\gamma) = \sqrt{2\pi}$$

Therefore, from (6) we have

$$c = (\sqrt{\gamma}/\sqrt{2}\pi)\ \exp(-\delta^2/2\gamma) \quad (7)$$

Replacing c into (4), the density function is

$$f(x) = (\sqrt{\gamma}/\sqrt{2\pi})\ \exp(-\delta^2/2\gamma)\ \exp(-\delta x-\gamma x^2/2) \quad (8)$$

which, finally, takes the form

$$f(x) = (\sqrt{\gamma}/\sqrt{2\pi})\ \exp((-1/2)\ (x+\delta/\gamma)^2\gamma) \quad (9)$$

We recognize from (9) that this is the normal density function with mean $\mu = (-\delta/\gamma)$ and variance $\sigma^2 = (1/\gamma)$. If $\gamma = 1$ and $\delta = 0$, then we have the standard normal density.

3. CONCLUSIONS

The approach described above for deriving the normal density from a differential equation is an effective teaching tool and has the following advantages.

1. Geometric inspection leads to the discovery of the density by the student.
2. Demonstration of the role of parameters can effectively be given.
3. Derivation of the mathematical form of density is shown to the student versus a presentation of the density.
4. To fit the needs of an introductory statistics course, this procedure lends itself hansomely by solving a simple differential equation.

REFERENCES

Gauss, C. F. (1809), Theoria Motus Corporum Coelestium.

Griffin, P. M., Smith, J.T. and Watts, G. D. (1982) «Deriving the Normal and Exponential Densities using EDA Techniques», The American Statistician 36, 4.

Laplace, P. S. (1812), Théorie Analytique des Probabilités, Paris

Whittaker, E.T. and Robinson, G. (1926). The calculus of Observations, London : Blackie.