TOWARDS A GENERAL WATER BALANCE MODEL

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SUMMARY

In this paper a general stochastic model is developed for the description of the water balance in an area. Both the water intake and output are considered as stochastic variables. The model, which naturally accommodates the basic features of the phenomenon, helps in evaluating quantities such as the distributuon of the wet period following some time instant. The application of the model in a specific case is examined.

1. INTRODUCTION

Although many climatic variables interact with the crop in many ways, rainfalls is the most important limiting factor in agricultural planning. Stochastic processes have been employed extensively for the analysis of rainfall occurrences. Early work described the distribution of wet and dry spell lengths (Lawrence, 1954 ; Cooke, 1953 ; Williams, 1952). The distributions fitted have been found to give reasonable fits at different sites (Green, 1970 ; Singh et al, 1981). A renewal theory approach was used in J.R. Green 1967 and 1964 and in Buishand 1977. Markov chain models have a large literature on rainfall modelling. The occurrence of rainfalls at Tel Aviv was analysed by fitting a two state first order Markov chain (Gabriel and Neumann, 1962). A seven state chain, each state corresponuing to a different amount of rain was analysed (Haan et al, 1976). Markov chains of higher than first order have been used as well (Katz, 1977 ;

Lung et al, 1977). Stern and Coe, 1984 use non - stationary Markov chains. The Literature on rainfall modelling is reviewed by Waymire and Gupta (1981) and Stern and Coe (1984). An important feature for agriculture is the time of the end of the wet season. If it occurs too soon the crop may not reach maturity while excessive wet weather may prevent ripening. The end of the growing season is the date when the soil water profile is too dry so that the growth of the crop to continue. This date can be evaluated by considering models which except from taking into consideration the pattern and amount of rainfalls will consider as well évapotranspiration (plus possibly runoff and drainage). The évapotranspiration can be evaluated as a function of various climatic variables (Doorembos and Pruit 1977 ; A. Ben Harrath et al, 1985 ; Dautrebande-Gaspar S. et al, 1983) which emphasises the importance for agriculture of variables other than rainfalls. Such models are called water balance models. They are essentially book-keeping procedures which estimate the balance between the income of water from precipitation and the outflow of water from évapotranspiration, runoff and drainage. They have been used mainly for computing the seasonal and geographic patterns of irrigation demand.

Existing water balance models (Cocheme and Franquin, 1967 ; Hills and Morgan, 1981 etr) incorporate actual rainfall data in different years together with average evaporation figures. The probability distribution of chavacteristies of the water balance is then a reflection of the year to year variability of the rainfall. Kottegoda et al, 1980 derive a rain all- runoff model using pulses and a transfer function, which subsequently extend. Alley, 1984, assesses the value of five regional water balance models to transform monthly precipitation and monthly potential évapotranspiration data to monthly and annual runoff estimates, using fifty years records O_1 monthly strearmlow in New Jersey. Among others it was found that all the models did well in reproducing annual flows and less well in simulating monthly flows.

The objective of this paper is to demonstrate the viability of an alternative approach which for a given site forecasts the end of the wet period. Using an initial water soil profile, the distribution of the intervals between rainfalls and information on évapotranspiration (ftd possibly drainage and runoff) evaluates the distribution of the remaining wet period following a time instant tt. It has the additional advantage over other models that it can take into consideration the changing pattern of rainfalls through the year. This is important since it is known (Stern and Coe, 1984) that the assumption of stationarity is not valid even for periods as short as one month. It differs as well from previous models in that it treats évapotranspiration as a stochastic variable. It is a first attempt for the development or a generalized water balance model which naturally accommodates the basic characteristics of the system such as ; intervals between rainfalls, amount of water in rainfalls, evaporanspiration (plus possibly runoff and drainage), initial water Soil profile.

2. THE MODEL

The supply of water to crops is directly influenced by the readily available soil water reserve. Rainfalls keep soil wet till the the rainfall water vanishes due to évapotranspiration (plus possibly runoff and drainage).

Evapotranspiration is dependent on several parameters such as the vegetative cover, the amount of soil water in the root zone, the vegetative state of the crop, the physical properties of the soil, and the evaporative power of the atmosphere Here it is assumed that it is a random variable with known pdf (df) $g_x(x)$ $(G_x(x))$. Therefore each rainfall keeps soil wet for a certain time period ; during this period the water of the rainfall is usefull and after this the soil profile becomes too dry for growth to continue.

In the sequel rainfalls are considered as instant events that is, all the water in the rainfalls joins the soil at the beginning of the rain. This assumption seems realistic due to the short time duration of rainfalls in relation to the long time periods this model examines.

Several ways have been used to model the behaviour of rainfall amounts. For example Smith and Schreiber (1974) fitted a theoretical distribution, Cole and Sherrif (1972) fitted an empirical distribution, Hiemstra and Greese (1970) generated rainfall amounts for shorter periods and summed them together to form daily totals. Stern and Coe (1984) fitted gamma distributions. In this work it is assumed that rainfall amounts follow a pdf (df) $g_3(x)$ $(G_3(x))$ and that the rainfall amounts of different rainfalls are independent. It is assumed as well that the time periods between successive rainfalls are independently and identically distributed random variables with pdf (df) $g_2(x)$ $(G_2(x))$.

Let assume that for every time instant t, WP(t) denotes the remaining wet period, that is the wet period following instant t. Consider that WP(t) has distribution function

$$F(x,t) \triangleq Pr(WP(t) \leqslant x)$$

Let RF(t) denote the number of rainfalls at time t, the water of which is still usefull. Then the distribution function and the mean value of WP(t) are given respectively by :

$$F(x,t) = \int_a Pr(WP(t) \leqslant x \mid RF(t) = a)\ Pr(RF(t) = a)da \qquad (1)$$

and

$$E(WP(t)) = \int_a E(WP(t) \mid RF(t) = a)\ Pr(RF(t) = a)da \qquad \textbf{(2)}$$

Relations (1) and (2) above permit us to evaluate the distribution of the wet period following an instant t as well as its mean value. In order to do so, assumptions have to be made regarding the distribution of evapotraspiration, interarrival times between rainfalls and rainfall amounts. In the next section it Will be attempted to work out analytically the above relations, as an example, in cases where g_1 (x), g_2 (x) and g_3 (x) are known distributions. But in cases where (1) and (2) could not be woked out analytically, it is believed that numerical intergration and modern fast compurets will suffice to get answers with reasonable accuracy.

3. APPLICATION OF THE MODEL

Assume that évapotranspiration is constant and equal to a mm per day and that the height of the water of a rainfall follows the negative exponential distribution with pdf

$$g_3(h)=b\ e^{-bb}\ ;\ h>0,\ b>0$$

If the random variable X represents the number of days the soil is kept wet due to a rainfall the water of which follows $g_3(h)$, then X has distribution function

$$G(x) = 1 - e^{-abx}$$

So X is negative exponentially distributed random variable with mean ab.

Since a rainfall keeps soil wet for a length of time X_1 of exponential duration, if RF(t) = η (n > 1) then

$$WP(t)=X_1+X_2+\ldots+x_n$$

where $X_1,\ X_2,\ldots,X_n$ are identically and independently distributed random variables with mean a b = 1/d.

We assume that $g_2(x)\ — c\ e^{-cx},\ x>0,\ c>0$

Let Pn(t) = Pr (RF(t) = n) for n= 0,1,2,...

According to the theorem of total probability we get the following expression for F(x,t)

$$F(x,t) = \sum_{n=0}^{\infty} Pr(WP(t) \leqslant x \mid RF(t) = n) \quad Pr(RF(t) = n)$$

$$= Pr(WP(t) \leqslant x \mid RF(t) = 0) \; P_0(t) + \sum_{n=1}^{\infty} Pr\left(\sum_{i=1} X_i \leqslant x\right) Pn(t)$$

$$= P_0(t) + \sum_{i=1}^{\infty} Pr \quad \left(\sum_{i=1}^{n} X_i \leqslant x\right) P_n(t)$$

The random variable follows the Erlang distribution with parameters n, d, so :

$$F(x,t) = P_0(t) + \sum_{n=1}^{\infty} \quad P_n(t) \int_{o}^{x} (d^n / (n-1)\,!\,) \; e^{-dy} \; y^{n-1} dy$$

where $P_n(t)$ is given bellow.

The mean value of WP(t), due to the theorem of total probability for mean values is given by

$$E\,(WP\,(t)\,) = \sum_{n=0}^{\infty} E(WP(t) \mid RF\,(t) = n \,.\, Pr(RF(t) = n)$$

$$= \sum E(X_1+X_2+ \dots +X_n)\ P_n(t)$$

$$= (1/d) \sum nP_n(t)$$

$$= (1/d)\ E(RF(t))$$

Since each rainfall keeps soil wet for a length of time of exponential duration which is independent of one another, $P_n(t)$ is given by (Cox. D.R. and Smith. R.L. 1961)

$$P_n(t) = e^{-(c+d)t} \Big[p^{(n-i)/2}\ I_{n-i}(at) + p^{(n-i-1)/2}\ I_{n+i+1}\ (at) + + (1-p)p^n \sum_{j=n+i+2}^{\infty} p^{-j/2}\ I_j(at) \Big]$$

where

$$a = 2d(c/d)^{0 \cdot 5}, \qquad p = c/d$$

$$I_n(x) = \sum_{n=0}^{\infty} ((x/2) / \ (n+m)!\ m), \qquad n \geqslant -1$$

is the modified Bessel function, and RF(O) = i, that is the number of rainfalls in the soil the water of which is usefull at the start of the period under examination is i.

4. CONCLUSIONS

In this work a general water balance model is developed which naturally accommodates the basic characteristics of the system such as ; intervals between rainfalls, amount of water in rainfalls, évapotranspiration (plus possibly runoff and drainage), initial water soil profile. The difference of this model over others is that it treats évapotranspiration as a random variable and that it evaluates the distribution of the end of the wet season analytically. The model should be further investigated by considering as well other distributions for évapotranspiration and the amount of water in the rainfalls. The application of the model to real data should be examined as well.

5. REFERENCES

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