ON THE DISTRIBUTION OF THE PRESENT VALUE OF AN INVESTMENT UNDER CERTAIN TIMING

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i, INTRODUCTION

In the analysis of any capital investment problem two factors arise for which descriptive methods must be developed. First the return and second the risk or uncertainty associated with that return. The problem of analyzing the magnitude and timing of returns is generally resolved by using the net present value. However, the treatment of risk is not entirely satisfactory.

A review of the capital budgeting literature gives a variety of techniques for evaluating risk. These techniques can be categorized as either deterministic or probabilistic approaches to risk analysis.

The payback period method is one of the earliest deterministic methods for evaluating risk. The use of this method is based upon the assumption that the shorter the payback period, the less risk the investment. The application of this method to the analyiss of risky investments assumes that an investment has assured net cash flows during its payback period but that its subsequent returns are so uncertain that they must be regarded as virtually nonexistent. From the fact that this method is not based upon the time value of money and does not take into account cash flows beyond the payback period or give any consideration to the uncertainty of future cash flows it is considered as an inadequate indicator of risk and a crude measure for reaching sound investment decisions.

The deterministic net present value method with the discount rate increased by an arbitrary amount for risk is another method of investment analysis for conditions of uncertainty. The main difficulties with this method are determination of the adjustment that should be made in the discount rate, and the use of single value estimates for future cash flows by discounting at a risk adjusted discount rate, is generally considered to be better than the payback period method.

Another approach to risk analysis is the certainty - equivalent method which uses discounting at a risk free interest rate and accounts for uncertainty in future cash flows by multiplying the forecasted cash flows for each period by a risk adjustment coefficient. Specifying these certainty equivalent coefficients for each period in the analysis is one of the primary drawbacks associated with this method.

Sensitivity analysis is a method that examines the effects on the measure of merit of the investments to variations in the key economic elements. Typically minimum, most likely, and maximum estimates of the major factors are combined to obtain extreme values of the measure, thereby estimating the total rang, of possible outcomes. This method can be an effective technique when used in conjuction with other methods for risk analysis. Hillier *[6]* has pointed out that this method should not be used alone since it cannot adequately asses the overall risk of an investment.

Each of the above methods uses «single - value» or «best» estimates for the magnitude and timing of returns. Such estimates do not clearly show the uncertainty inherent in future returns. In recent years several methods have been developed which specifically deal with the uncertainty surrounding the future cash flows. These techniques use probability and distribution theory and in some cases with the help of modern computers provide the decision make with much more information on which to base his investment decisions. Principal among the probabilistic techniques are the analytic methods developed by Hillier [6,7]. These methods make use of the properties of statistical distributions and, under certain assumptions, derive the distribution function of the two major profitability criterion functions [Net Present Value and internal Rate of Return] from the estimated mean and variance of the individual cash flow for each time - perod. Such procedures form an approach for the evaluation of single risky investments utilizing discrete cash flows and discrete discounting. This technique was extented to the case of serveral interrelated investments by Hillier [8] and refined by Wagle [15]. It was further developed to consider continuous cash flows with continuous discounting by Motazed [10]. The element of uncertain timing of the cash fows has been given consideration by Motazed [10], Perrakis and Henin [12] and Young and Contreras [18]. Reismann and Rao [13] have studied a stochastic rate of inflation independeut to the discounting rate. Motazed [10] and Zin, Lesso, and Motazed [20]utilized cash flow profiles developed by Young [17] and Laplace transform methods discussed by Buck and Hill [2,3]. Grubbstrom [5] and Young [17] to develop analytic expressions for the expected value, variance and semi - variance of the pressent value. Canada and Wadsworth [4] not only establish a methodology for approximating the expected value and variance of present value but also present a method for evaluating two comparable projects based on their distributions of present value. Tanchoco and Buck [14] present a closed form methodology based upon the Zeta transform for obtaining statistical moments of the net present value for discrete cash flows. Zinn [19] demonstrated the advantage of using moment generating functions, to obtain the formulae developed by Motazed.

The above analytic probabilistic techniques, with the exclusion of Tanchoco and Buck [14] and Perrakis and Henin [12]' center about the derivation of mean value, variance and, in some cases, semivariance of the net present value for an fnvestment project. These values are then used to estimate the risk associated with the return. Tanchoco and Buck include the third and fourth moments of the distribution of the net present value of discrete cash flows. They also indicate the ability to calculate higher moments. Only Perrakis and Henin mention the possibility of attenting to determine, at least numerically, the actual distribution function of the net present value is preferable in view of the fact that a finite set of moments does not ' necessarily specify a unique distribution function from which the moments arose.

This paper presents the results for the evaluation of the distribution function of the present value of an investment, in which the cash flows take place at discrete equidistant time periods, and the investment terminates after a given number η of periods. The initial cash outlay is deterministic and the cash flows are continuous independend random variables with known distribution functions and the discount rate is given. These assumptions are quite plausible, because in many investrnemt projects the initial outlay is known in advance and the cash flows are random variables.

2. Results

In this section certain results for the distribution function of the present value of an investment project are established. We consider investments with cash flows,

[X_k : k= 1, 2,. . . .,n) at the and of years 1,2,. . . ., η respectively, where X_k are continuous independent randon variables. The economic life of the investment and the discount factor are not considered as random variables. Hence our pur. pose is to find, under certain assumptions for the cash flows, the distribution function of the linear statistics

$$Y = \alpha X_1 + \alpha^2 X_2 + \dots + \alpha^n X_n$$

where $\alpha = 1/(1 + r)$ and r is the rate of interest.

Proposition 2.1

If the cash flows [X_k : k= 1,2,. . . . ., n) are independent exponentially distributed random variables with parameters $\lambda_k = \lambda\alpha^k$, k = 1,2,. . . . . .,η $\lambda > 0$, then the present value Y is a gamma distributed random variable with scale parameter λ and index parameter n.

Proof

The characteristic function of $a^k X_k$, k= 1,2,. . . . . ., is given by

$$\varphi^k(u) = \frac{\lambda_k}{\lambda_k - i\alpha^k u} \tag{2.1}$$

Hence the characteristic function γ(u) of present value Y has the form

$$\gamma(u) = \prod_{k=1}^{n} \left(\frac{\lambda_k}{\lambda_k - i\alpha^k u}\right) \tag{2.2}$$

Letting $\lambda_k = \lambda a^k$ in (2.2) we get that

$$\gamma(u) = \left(\frac{\lambda}{\lambda - iu}\right)^n$$

Hence the present value Y is a gamma distributed random variable with scale parameter λ and index parameter n.

Remark 1 to Proposition 2.1

We suppose that the cash flows $[X_k : \mathbf{k} = 1,2,\ldots,n)$ are independent gamam distributed random variables with scale parameters $\lambda\varkappa == \lambda a^k$, $\mathbf{k} = 1,2,\ldots,n$, λ L 0 and index parameters Ck >0. From proposition 2.1 it easily follows that the characteristic function γ(u) of Y has the form

$$\gamma(u) = \left(\frac{\lambda}{\lambda - iu}\right)C$$

where $c = c_1 + c_2 + \ldots + c_n$.

Remark 2 to Proposition 2.1

The function

$$\frac{\lambda}{\lambda + | u |}, \ \lambda > 0,$$

is the characteristic function of a power mixture of Cauchy distribution. With exponential mixing distribution, we suppose that the cash flows $[X_k : k— 1,2,\ldots,n)$ are independent random variables with characteristic function

$$\varphi_k |u| = \frac{\lambda_k}{\lambda_k + [u]}, \quad \lambda_k > 0$$

If $\lambda x = \lambda a^k$ then from proposition 2.1 it follows that the characteristic function $\gamma(u)$ of Y has the form

$$\gamma(u) = \left(\frac{\lambda}{\lambda + |u|}\right)^n$$

Remark 3 to Proposition 2.1

The function

$$\int_0^\infty \frac{\lambda}{\lambda - iux} dF(x), \quad \lambda > 0$$

is the characteristic function of a scale mixture of exponential distributions with mixing distribution F(x). We suppose that the cash flows [Xk:k=1,2,.....,n) are independent random variable with characteristic function

$$\varphi_k(u) = \int_0^\infty \frac{\lambda k}{\lambda k - iux} dF(x), \quad \lambda_k > 0.$$

If $\lambda_k = \lambda a^k$ then from proposition 2.1 it follows that the characteristic function $\gamma(u)$ of Y has the form

$$\gamma(u) = [\int_0^\infty \frac{\lambda}{\lambda - iux} dF(x)]^n.$$

Proposition 2.2

If the cash flows [Xk :k = 1,2,. . . . ,n) are independent Laplace distributed randon variables with paramètres $\lambda\varkappa = \lambda\alpha_2$ k= 1,2,...., η λ) 0, then the present value Y is a random variable distributed as the nth convolution of the Laplace distribution with parameter λ.

Proof

The characteristic function of the random variable $a^k Xk$, k= 1,2,. . . . ,n is given by

$$(\varphi>_k(u) = \frac{\lambda_k}{\lambda_k + a2k_u2}$$

Hence the characteristic function of Y has the form

$$\gamma(u) = \prod_{k=1}^{\eta} \left(\frac{H}{\lambda k + \alpha 2tu2}\right) \tag{2.3}$$

Letting $\lambda_k = \lambda\alpha^{2k}$ in (2.3) we get that

$$\gamma(u) = \left(\frac{\lambda}{\lambda + u^2}\right)^n.$$

Hence the random variable Y is distributed as the nth convolution of the Laplace distribution with parameter λ.

Remark 1 **to Proposition** 2.2

We suppose that the cash flows [Xk : k— 1,2,. . . . . ,n) are independent random variables having characteristic function

$$\varphi_k(u) = \left(\frac{\lambda_k}{\lambda_k + u^2}\right)^{c_k}, \quad \lambda_k > 0$$

where $\lambda_k = \lambda a^{2k}$, k= 1,2,...., n, $\lambda > 0$. From proposition 2.2 it easily follows that the characteristic function $\gamma(u)$ of Y has the form

$$\gamma(u) = \left(\frac{\lambda}{\lambda + u^2}\right)^{c}$$

where $c = c_1 + c_2 + \dots + c_n$.

Proposition 2.3

If the cash flow (X_k : k= 1,2,...,n) are independent normally distributed random variables with mean μ and variance σ^2 then the random variable

$$\lim_{n \to \infty} Y$$

is normally distributed with mean

$$\frac{\alpha \mu}{1 - \alpha}$$

and variance.

$$\frac{\alpha^2 \sigma^2}{1 - \alpha^2}$$

Proof

The characteristic function of the random variable $\alpha^k X_k$, k=1,2,....,n is given by

$$\varphi_k(u) = \exp\{i\mu\alpha^k u - \frac{1}{2}\alpha^{2k}\sigma^2u^2\}. \qquad (2.4)$$

Hence the characteristic function $\gamma(u)$ of Y has the form

$$\gamma(u) = \prod_{k=1}^{n} \exp\{i\mu\alpha^k u - \frac{1}{2}\alpha^{2k}\sigma^2u^2\}.$$

$$= \exp\{i\mu u \sum_{k=1}^{n} \alpha^k - \frac{1}{2}\sigma^2u^2 \sum_{k=1}^{n} \alpha^{2k}\}$$

$$= \exp\{i\mu\alpha \frac{1-\alpha^n}{1-\alpha} u - \frac{1}{2}\sigma^2\alpha^2 \frac{1-\alpha^{2n}}{1-\alpha^2} u^2\}.$$

Since $0 < \alpha < 1$ it follows that

$$\lim_{n\to\infty} \frac{1-\alpha^n}{1-\alpha} = \frac{1}{1-\alpha}$$

and

$$\lim_{n\to\infty} \frac{1-\alpha^{2n}}{1-\alpha^2} = \frac{1}{1-\alpha^2}$$

Hence from (2.5) we get that the random variable $\lim_{n\to\infty} Y$ is normally distributed with mean

$$\frac{\alpha\mu}{1-\alpha}$$

and variance

$$\frac{\alpha^2\sigma^2}{1-\alpha2}$$

Proposition 2.4

If the cash flows (Xk : k — 1,2,....,n) are independent Cauchy distributed randon variables with location parameters μ and scale parameters λ then the randon variable

$$\underset{n->oo}{\text{ban}}\ Y$$

is Cauchy distributed with location parameter

$$\frac{\alpha\mu}{1-\alpha}$$

and scale parameter

$$\alpha\lambda$$

$$1-\alpha$$

Proof

The characteristic function of the random variable a^kXk, k= 1,2,. . . . ,n is given by

$$q>k(u) = \exp\{a^k[i\mu u - \lambda \mid u \mid \ \backslash. \tag{2.6}$$

Hence the characteristic function γ(u) of Y has the form

$$\gamma(u) = \prod_{k=1}^{n} \exp\{\alpha^k[i\mu u - \lambda\,|u|\,]\}$$

$$= \exp\{[i\mu u - \lambda|u|] \sum_{k=1}^{n} \alpha^k\}$$

$$= \exp\{\alpha[i\mu u - \lambda|u|] \frac{1-\alpha^n}{1-\alpha}\} \tag{2.7}$$

Since $0 < \alpha < 1$ from (2.7) it follows that the random variable

$$\lim_{n \to \infty} Y$$

is Cauchy distributed with location parameter $\alpha\mu\,(1-a)$ and scale **parameter** $\alpha\lambda\,(1-\alpha)$.

Proposition 2.5

Let the cash flows $(X_k : k = 1,2,\ldots,n)$ be independent distribution random variables with characteristic functions

$$\varphi_k(u) = \exp\{\int_0^1 \frac{w_k(ux) - 1}{x}\,dx\},\ k = 1,2,\ldots,n \tag{2.8}$$

where $W_k(u)$ is the characteristic function of a distribution function $W_k(x)$ on $[0, \infty)$ with finite mean. Then the characteristic function $\gamma(u)$ of present value Y is of the form (2.8).

Proof

The characteristic function of $a_k X_k$, k — 1,2,....» n is given by

$$\varphi_k \alpha^k) = \exp\{ \int_0^1 \frac{w_k(\alpha^k ux) - 1}{x} dx \}.$$

Since

$$\psi(u) = \sum_{k=1}^{\eta} \frac{1}{\eta} w_k(a^k u)$$

is also, the characteristic function of a distribution function on [0, <x) with finite mean and

$$\gamma(u) = \exp\{ \int_0^1 \frac{\Psi(u\chi) - 1}{x} dx \},$$

it follows that the characteristic function $\gamma(u)$ of the present value Y is of the form (2.8).

Remark 1 to Proposition 2.5

Characteristic functions of the form (2.8) belong to class L. Distribution functions with characteristic functions belonging to class L are unimodal [16]. Since the characteristic function $\gamma(u)$ of the present value Y is of the form (2.8)it follows that the distribution function of the present value is unimodal.

The concept of the unimodality, besides being useful and well * known in probability and statistics, is basic to many practical problems. In economics, unimodality helps to obtain better statistical inferences,

Proposition 2.6

Let the cash flows $(X_k : k = 1,2,\ldots,n)$ be independent random variables with distribution functions belonging to class U. Then the distribution function **of** the present value belongs to class U.

Proof

The characteristic function of $\alpha^k X_k$, $k = 1,2,\ldots,n$ is given by

$$\varphi_k(\alpha^k u) = \exp\left\{ \int_0^1 \log\psi_k(\alpha^k ux)dx \right\}$$

where $\psi_k(u)$ is an infinitely divisible characteristic function [11]. Hence the characteristic function $\gamma(u)$ of the present value can be written in the form

$$\gamma(u) = \prod_{k=1}^{n} \exp\left\{ \int_0^1 \log\psi_k(\alpha^k ux)dx \right\}$$

$$= \exp\left\{ \sum_{k=1}^{n} \int_0^1 \log\psi_k(\alpha^k ux)dx \right\}$$

$$= \exp\left\{ \int_0^1 \sum_{k=1}^{n} \log\psi_k(\alpha^k ux)dx \right\}$$

$$= \exp\left\{ \int_0^1 \log \prod_{k=1}^{n} \psi_k(\alpha^k ux)dx \right\}.$$

The function

$$\theta(u) = \prod_{k=1}^{n} \psi_k(\alpha^k u)$$

is an infinitely divisible chararistic function. Since

$$\gamma(u) = \exp\left\{\int_0^1 \log\theta(ux)dx\right\}\cdot$$

it follows that the distribution function of the present value Y belongs to classU.

Remark 1 to Proposition 2.6

Distribution functions of class Uwhich are (0) symmetrical are also (0) unimodal [9]. The convolution of (0) symmetrical and (0) unimodal distribution functions is also an (0) symmetrical and (0) unimonal distribution function [9]. Hence from proposition 2.6 it follows that if the distribution functions of the cash flows are (0) symmetrical of class U then the distribution function of the present value Y is (0) symmetrical (0) unimodal of class U. This means that the result of proposition 2.6. helps the decision maker to obtain better statistical inferences.

Remark 2 to Proposition 2.6

Let the cash flows (X_k : k= 1,2,.. . ., n) be independent random variables with distribution functions belonging to class Up introduced by Artikis [1]. Following proposition 2.6 we can easily prove that the distribution function of the present value belongs to class Up.

3 CONCLUSIONS

The paper presents certain results for obtaining the distribution function of the present value of an investment project. Through the use of characteristic functions the distribution function of the present value has been obtained for several conceptual examples and new results regarding the distribution of present value cash flows obtained which can be used in evaluation of risky investment and projects. Applications of the theory developed here will be presented in later papers.

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