NON-EXISTENCE OF A SOLUTION IN A FINITE HORIZON CONTINUOUS MODEL

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We consider here the question of existence of a solution in a finite horizon, continuous model of optimal growth. Infinite horizon models with no solution have been discussed in the literature and there has also been some discussion of finite horizon models.

For the case of infinite horizon models intuitive explanations for the lack of a solution have, in some cases, been provided. The models might be such that we require to be able to slice a fixed quantity into equal parts and distribute it over an infinite horizon, which is impossible. Or they might be such that the unboundedness of the felicity function and the production function, and the lack of a positive rate of pure time preference imply that consumption is postponed forever (for example, [1]). For models with these characteristics, the lack of a solution does not depend on whether the formulation is that of discrete or continuous time.

The following are examples of infinite horizon models without a solution.

(i) The optimal consumption of a stock.

Continuous Model 1

Maximize $\int_0^\infty \log c(t)\, dt$

Subject to

$$\int_0^\infty c(t)\, dt = W_0$$

$$c(t) \geq 0, \quad W_0 > 0$$

Discrete Model 1

Maximize $\sum_{i=0}^\infty \log c_i$

Subject to

$$\sum_{i=0}^\infty c_i = W_0$$

$$c_i \geq 0, \quad W_0 > 0$$

where W_0 is the stock of a good at time 0 and $c(t)$, c_i consumption at time t and i.

Now it is easy to see that if instead of an infinite horizon we had a finite horizon the unique solution would have been

$$c(t) = \frac{W_0}{T} \qquad \text{and} \qquad c_i = \frac{W_0}{T+1}$$

for the continuous and discrete models respectively. On the other hand as $T \Rightarrow \infty$ we obtain the worst possible path of zero consumption and hence no solution exists for either problem.

The reason is that although we can divide W_0 in equal parts over a finite period we cannot do so when the horizon is infinite. The attempt to obtain such a division results in the worst possible path of zero consumption.

(ii) A model with production.

Continuous Model 2

Maximize $\int_0^{\infty} \log c(t)\, dt$

Subject to $c(t) + \dot{k}(t) = bk(t)$

$c(t) \geq 0, \quad k(t) \geq 0$

$k(0) = W_0 > 0$

Discrete Model 2

Maximize $\sum_{i=0}^{\infty} \log c_i$

$c_0 + k_0 = W_0$

$c_1 + k_1 = bk_0 + k_0$

....

$c_i + k_i = bk_{i-1} + k_{i-1}$

$c_i \geq 0, \quad k_i \geq 0, \quad W_0 > 0.$

where W_0 is the initial level of capital stock, the positive constant b is the output – capital ratio, $c(t)$, c_i are consumption at time t and i and $k(t)$, k_i capital stock at time t and i.

Now for finite horizon T the unique solution is

$$k(t) = \left(k_0 - \frac{k_0}{T} t\right) e^{bt} \qquad k_i = \frac{k_0}{T}(T-i)(1+b)^i$$

$$c(t) = \frac{k_0}{T} e^{bt} \qquad c_i = \frac{k_0}{T}(1+b)^i$$

$$k_0 = \frac{T}{T+1} W_0$$

for the discrete and continuous models respectively.

On the other hand as $T \Rightarrow \infty$, $c(t)$ and c_i tend to zero. The fact that the felicity function, $\log c(t)$, and the production function, $bk(t)$, are unbounded implies that consumption is postponed in terms of the infinite future which never comes. There is no terminal period over which the benefits from the earlier postponement of consumption can be reaped and hence there is no solution to either problem.

For the case of finite horizon continuous models conditions for the existence of optimal paths have been discussed in [2]. The discussion here is intended to supplement the analysis there.

Of cource, finite horizon models of a discrete formulation will always have a solution. This is due to Weierstrass theorem, that a continuous function on a compact set attains its maximum. One can then trace the lack of solution in a finite horizon continuous model to the fact that the conditions of the Weierstrass theorem are not satisfied.

A continuous model can also be viewed as the limit of a sequence of discrete models. This enables us to obtain an intuitive explanation why in some finite horizon continuous models a solution fails to exist. The limit of the solutions of the models in the sequence is not available.

The discussion here is in the context of a simple model. However, it will be evident that a similar explanation can be provided in more general models.

Consider the following optimal growth models:

Continuous Model 3

Maximize $\int_0^T c(t)\,dt$

Subject to

$c(t) = bk(t) - \dot{k}(t)$

$c(t) \geq 0 \quad k(t) \geq 0$

$k(0) = W_0, \quad k(T) = 0$

Discrete Model 3

Maximize $c_0 + c_1 + c_2 + \ldots + c_{T-1}$

Subject to

$c_0 + k_0 = W_0$

$c_1 + k_1 = bk_0 + k_0$

$c_2 + k_2 = bk_1 + k_1$

....

$c_{T-1} = bk_{T-2} + k_{T-2}$

$c_i \geq 0, \quad k_i \geq 0$

where c denotes consumption, k capital stock and W_0 is a positive constant. In the discrete model, c_{T-1}, the quantity of the good allocated to consumption at $T-1$ is to last for the period $[T-1, T]$ and hence the length of the planning horizon is in both cases T.

Consider first the discrete model. Irrespective of the value of b, it has a solution. For $b<0$ the solution is $c_0 = W_0$ and $c_i = 0$ for $i = 1, 2, \ldots, T-1$.

For $b=0$ any vector of c_i's which satisfies the constraints, is a solution. For $b>0$ the solution is $c_i = 0$ for $i = 0, 1, \ldots, T-2$ and $c_{T-1} = W_0 (b+1)^{T-1}$.

These solutions are explicitly derived in Appendix I.

Assume now that the time period between decisions becomes shorter and shorter. As we truncate the intervals further and further, the same kinds of solutions are obtained. Of cource the shorter intervals will imply that for $b<0$ the quantity W_0 in the first period and for $b>0$ the accumulated quantity of the good in the last period must now be consumed at a faster rate. The limit of these rates, as the distance between decision points tends to zero, is infinity. As we shall see, this has implications for the solution of the continuous model.

Next we consider the continuous model. Application of the Euler equation of calculus of variations implies that we require

$$b + \frac{d}{dt} 1 = 0$$

For $b=0$ the Eulerian relation imposes no restriction and any feasible path is a solution. On the other hand for $b \neq 0$ there is no solution.

We shall now view Continuous Model 3 as the limit of a sequence of discrete models, as the interval between decisions tends to zero. This will provide an intuitive explanation why a solution fails to exist.

In Discrete Model 3 above intevals are of unit length and one possible interpretation of this model is as follows. A quantity of the good, c_t, is allocated to consumption at the beginning of period t and it is to be consumed at a constant rate throughout the period. This constant rate of consumption yields a constant utility rate, $u(c_t)$, which integrates over the unit interval to $u(c_t)$. In our model $u(c_t) = c_t$.

Now suppose that decisions are taken at intervals of half the time unit, rather than at intervals of one time unit. In order to allow for comparisons, c_t denotes again consumption per unit of time. A typical production constraint is now

$$\frac{1}{2} c_{t+\frac{1}{2}} + k_{t+\frac{1}{2}} = \frac{1}{2} bk_t + k_t.$$

The contribution of this consumption decision to the utility sum is $\frac{1}{2} c_t$.

For the case when the interval between consumption decisions is Δt Discrete Model 3 is replaced by

$$\text{Maximize} \quad \sum_{i=0}^{\frac{T}{\Delta t}-1} c_i \, \Delta t \qquad t = i\Delta t$$

Subject to

$$c_0 \Delta t + k_0 = W_0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$c_{t+\Delta t}\, \Delta t + k_{t+\Delta t} = bk_t \, \Delta t + k_t$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$c_{\frac{T}{\Delta t}-1}\, \Delta t = bk_{\frac{T}{\Delta t}-2}\, \Delta t + k_{\frac{T}{\Delta t}-2}$$

$$c_i,\ k_i \geq 0$$

Consider now the case $b<0$. Irrespective of the value of Δt the solution will be to consume everything in the first period and $c_0 = \frac{W_o}{\Delta t}$. The consumption rate depends on Δt and $\lim_{\Delta t \Rightarrow 0} c_0 = \infty$. Hence in the limit no solution exists. On the other hand as $\Delta t \Rightarrow 0$ we capture the continuous time problem. Therefore we have provided an intuitive explanation why for $b<0$ the continuous model has no solution.

Consider next the case $b>0$. For given Δt the solution to Discrete Model 3 is $c_i = 0$ for $i = 0, \ldots, \frac{T}{\Delta t} - 2$ and $c_{\frac{T}{\Delta t}-1} = W_0 \, (b\Delta t + 1)^{\frac{T}{\Delta t}-1}$ and as $\Delta t \Rightarrow 0$ consumption is postponed for the last point in time T. This explains why for $b>0$ there is no solution to the continuous model. It is not possible to consume at an instant a finite amount $c_T = W_0 \, e^{bT}$ no matter how high the consumption rate is.

It is now evident why for $b \neq 0$ a solution to the continuous model will fail to exist. Infinite consumption rates are not available. Of cource for $b=0$ the situation is completely analogous to the one in the discrete case. Any feasible allocation of W_0 gives utility W_0.

Next we show that the values of the utility integral have as supremum W_0 when $b<0$ and $W_0 e^{bT}$ when $b>0$. Of course the supremum cannot be attained.

First consider the case $b<0$. It is easy to check that the following sequence of pairs of paths in feasible

$$k_n(t)=W_0-nt \qquad c_n(t)=(bW_0+n)-bnt \quad \text{for } t \leq \frac{W_0}{n} < T$$

$$k_n(t)=0 \qquad c_n(t)=0 \qquad \text{for } t > \frac{W_0}{n}$$

where integer $n \geq n_0$ and $bW_0+n_0>0$.

The implied sequence of values of the utility integral is

$$\int_0^{\frac{W_0}{n}} \{(bW_0+n)-bnt\}\, dt = [(bW_0+n)\, t - \frac{bnt^2}{2}]_0^{\frac{W_0}{n}} =$$

$$(bW_0+n)\frac{W_0}{n} - \frac{bn}{2}\,\frac{W_0^2}{n^2} = W_0 + \frac{b}{2}\,\frac{W_0^2}{n} < W_0$$

and tends to W_0 as $n \Rightarrow \infty$. On the other hand there does not exist a feasible pair which gives utility W_0. To see this consider any feasible path $c(t)=bk(t)-\dot{k}(t)$. The corresponding value of the utility integral is, for all T,

$$\int_0^T (bk(t)-\dot{k}(t))\, dt = \int_0^T bk(t)\, dt - k(T) + W_0 < W_0$$

since $b<0$.

Second consider the case $b>0$. The following pair

$$k(t)=W_0\, e^{bt} \qquad c(t)=0 \qquad \text{for } t<\tau$$

$$k(t)=(W_0\, e^{b\tau} - \frac{1}{b}\, \frac{W_0\, e^{b\tau}}{T-\tau})\, e^{b\,(t-\tau)} + \frac{1}{b}\, \frac{W_0\, e^{b\tau}}{T-\tau} \quad c(t)=\frac{W_0\, e^{b\tau}}{T-\tau} \quad \text{for } \tau \leq t \leq T$$

is for sufficiently large τ^* and for all $\tau \geq \tau^*$ feasible, i.e. $k(t) \geq 0$.

The corresponding utility integral is

$$\int_{\tau}^{T} \frac{W_0 e^{b\tau}}{T-\tau} \, dt = W_0 e^{b\tau}$$

and tends to $W_0 e^{bT}$ as $\tau \Rightarrow T$.

Next we show that there is no feasible path which gives utility $W_0 e^{bT}$. The value of the utility integral corresponding to feasible $c(t)$ is

$$\int_0^T (bk(t) - \dot{k}(t)) \, dt = \int_0^T bk(t) \, dt - k(T) + W_0 < -W_0 + W_0 e^{bT} - k(T) + W_0 \leq W_0 e^{bT}$$

The only case where the strict inequality above does not hold is when $c(t) = 0$ but then the value of the utility integral is zero.

Alternatively, Appendix II shows that if, for either $b < 0$ or $b > 0$, path $k(t)$ is proposed as optimal it is possible to find $k^1(t) \geq 0$ such that

$$\int_0^T b(k^1(t) - k(t)) \, dt > 0 \quad \text{and} \quad c^1(t) \geq 0.$$

As to a more formal mathematical explanation, lack of compactness of the relevant set in an appropriate topological space allows for the possibility that the utility integral does not attain a maximum.

Suppose the consumption paths are also required to be continuous and consider the space $C[0, T]$ of all continuous real valued functions on $[0, T]$ with norm $||f|| = \sup \{|f(t)| : t \varepsilon [0, T]\}$. $C[0, T]$ is a normed linear space and the utility integral is a bounded linear functional on this space and therefore continuous. On the other hand the set of feasible consumption paths is not compact in this space, i.e., not every sequence of feasible consumption paths converges to a feasible limit path.

We establish the lack of compactness by considering the feasible pair

$$k_n(t) = W_0 e^{-nt}, \qquad c_n(t) = (b+n) W_0 e^{-nt}$$

where integer $n \geq n_0 > b$. The sequence $< c_n(t) >$ has $||c_n|| = (n+b) W_0$ and therefore it does not contain a convergent subsequence.

Finally, lack of solution could appear also in more general formulations. For example, in the model above the presence of a discount factor $e^{-\delta t}$, with $\delta > 0$, would imply that a solution exists only for $b=\delta$. For $b<\delta$ it is not possible to consume W_0 at $t=0$ and for $b>\delta$ it is not possible to consume $W_0\,e^{bt}$ at $t=T$. Also a more general production function $f(k)$ with $f'(k)>0$ could have been used. Again the continuous model can be viewed as the limit of a sequence of discrete models and an explanation for the lack of solution can be provided. Infinite consumption rates are not available.

APPENDIX I

In this appendix we obtain explicitly the solution to Discrete Model 3 which is of course a linear programming problem with equality constraints. For concreteness we take $T-1=5$.

We have

Primal:

Maximize $c_0+c_1+c_2+c_3+c_4+c_5+ok_0+ok_1+ok_2+ok_3+ok_4$

Subject to

$$c_0+k_0=W_0 \qquad : p_0$$
$$c_1+k_1-(b+1)\,k_0=0 \qquad : p_1$$
$$c_2+k_2-(b+1)\,k_1=0 \qquad : p_2$$
$$c_3+k_3-(b+1)\,k_2=0 \qquad : p_3$$
$$c_4+k_4-(b+1)\,k_3=0 \qquad : p_4$$
$$c_5-(b+1)\,k_4=0 \qquad : p_5$$
$$c_i\geq 0, \quad k_i\geq 0$$

and

Dual:

Minimize $c=p_0\,W_0$

Subject to

$$p_0\geq 1, \quad p_1\geq 1, \quad p_2\geq 1, \quad p_3\geq 1, \quad p_4\geq 1, \quad p_5\geq 1$$
$$p_0-(b+1)\,p_1\geq 0 \qquad p_1-(b+1)\,p_2\geq 0$$
$$p_2-(b+1)\,p_3\geq 0 \qquad p_3-(b+1)\,p_4\geq 0$$
$$p_4-(b+1)\,p_5\geq 0$$

First we consider the case $b=0$. Any c_0, c_1, ..., c_5, with the corresponding k_0, k_1, k_2, k_3, k_4, which is feasible and sums up to W_0 is a solution to the primal. The solution to the dual has $p_0=1$ and all other p_i's can be set equal to 1.

Second we consider the case $b<0$. Now it does not pay to postpone consumption beyond the first period. The solution to the primal is $c_0=W_0$ and all other c_i's and all k_i's equal to zero. The solution to the dual has $p_0=1$ and other p_i's can be chosen to increase in a way that satisfies all constraints.

Finally we consider the case $b>0$. It now pays to postpone consumption up to the last period. The solution to the primal is $c_0=\ldots=c_4=0$, $c_5=(b+1)^5 W_0$, $k_t=W_0 (b+1)^t$, for $t=0, \ldots, 4$. The solution to the dual has $p_0=(b+1)^5$ and the other p_i's can be chosen to decrease in a way that satisfies all constraints.

For completness we mention that if the number of periods is infinite the case $b=0$ and $b<0$ still have an optimal c_t sequence. On the other hand $b>0$ implies that no optimal consumption sequence exists. Consumption gets postponed for ever.

APPENDIX II

In this appendix we show that, for $b \neq 0$, if $(c(t), k(t))$ is proposed as an optimal pair we can always construct feasible $(c^1(t), k^1(t))$ which gives a higher utility level. Hence the original assumption is incorrect and no optimal paths exist.

Let $c(t)=bk(t)-\dot{k}(t)$ be the path proposed as optimal and

$c^1(t)=bk^1(t)-\dot{k}^1(t)$ a feasible path which will be constructed to overtake $c(t)$.

Comparing the utility integrals we have

$$\int_0^T c(t)\,dt-\int_0^T c^1(t)\,dt=\int_0^T b(k(t)-k^1(t))\,dt-\int_0^T (\dot{k}(t)-\dot{k}^1(t))\,dt=$$

$$\int_0^T b(k(t)-k^1(t))\,dt$$

where the last equality follows form the fact that

$$\int_0^T (\dot{k}(t)-\dot{k}^1(t))\,dt=[k(t)-k^1(t)]_0^T=0$$

since both $k(t)$ and $k^1(t)$ start at W_0, and $k(T)=k^1(T)=0$, by assumption.

Now for path $c^1(t)$ to overtake path $c(t)$ we require

$$\int_0^T (k^1(t)-k(t))\,dt>0 \quad \text{if } b>0 \quad \text{and} \quad \int_0^T (k^1(t)-k(t))\,dt<0 \quad \text{if } b<0.$$

This is always possible to achieve in both cases. If $b>0$ consumption can be postponed initially and then in a shorter period the accumulated $k^1(t)$ can be consumed in such a way so that $\int_0^T (k^1(t)-k(t))\,dt>0$. On the other hand if $b<0$ the inequality $\int_0^T (k^1(t)-k(t))\,dt<0$ can be satisfied by keeping $k^1(t)$ below $k(t)$. At the same time we can have $c(t)$ feasible.

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