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OPTIMAL MONETARY POLICY IN A SIMPLE STOCHASTIC MONETARY MODEL WITH RATIONAL EXPECTATIONS

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1. INTRODUCTION

A récurrent problem in equilibrium monetary models with rational expectations is their inhérent instability. This feature of such models has been pointed out by, for example, Burmeister and Dobell (4), Olivera (12) and (13), Black (3), Sargent and Wallace (15) and Gertler (8); and this is by no means an exhaustive list.

Suggested "solutions" to this problem of instability have tended to fall into three main classes. A typical example in the first category is provided by Sargent and Wallace (15). Hère, it is assumed that rational agents "rule out" unstable paths. A "forward-looking" solution is suggested in which discontinuous jumps in the price level are permitted in response to changes in the exogeneously determined money supply policy. By this means the economy follows a séquence of rational expectations equilibria. Basically similar solution techniques hâve also been adopted by Lucas (11) and by Sargent and Wallace (16).

The second type of solution to the problem consists of building into the model some degree of sluggishness in the adjustment of priées and/or wages. Following Fischer (7), such rigidities are often explained as the resuit of the existence of long-term contracts. Examples of this type of model may be found in the papers by Gertler (8), Hoel (9) and Peel and Chappell (14). In each case a stability condition similar to that of Cagan is developed. Generally, such models are stable under exogenous money supply policies providing that the speed of adjustment of priées and/or wages is slow enough.

In the final category are the solutions discussed by Olivera (12), (13) and Black (3). The distinguishing feature hère is the distinction between active (by which is meant exogenous) and passive (i.e. endogenous) monetary policy. It has been shown that if money supply is determined endogenously, stability may almost always be ensured. A logical development hère would seem to be to enquire into the nature of an optimal monetary policy. That is the aim of this paper but we defer discussion of the choice of optimality criterion until section 3 below.

The structure of the paper is as follows. In section 2 our économie model is introduced and its reduced form, a stochastic differential équation, is derived. Section 3 deals with the stochastic optimization problem and dérives the optimal money supply policy. Finally, in section 4, some concluding remarks are offered.

2. A STOCHASTIC MONETARY MODEL

Our model closely follows that of Taylor (17). Two major différences, however, are our assumption of rational inflation expectations and the introduction of a measure of uncertainty. Uncertainty is incorporated in a relatively simple way by including an additive stochastic disturbance term in our description of the IS curve. There is no real loss in generality in assuming that the is curve is the only stochastic relationship. Stochastic terms could also be added to the LM curve and the augmented Phillips curve but this would resuit in composite error terms in the "reduced form" équations; confining uncertainty to the IS curve results in great économies of notation.

Investment demand is a decreasing function of the real rate of interest. Thus the IS curve is given by:

$$\ln.\ (Y) = a - br + \sigma W; \qquad a, b, \sigma > 0 \tag{1}$$

where Y is real income, r is the real rate of interest and W is a zero – mean continuous time white noise process. The ratio of desired real money balances to real income is a decreasing function of the nominal rate of interest. Therefore the LM curve is given by:

$$\ln.\ (M/PY) = \alpha - \beta i, \qquad \alpha, \beta > 0 \tag{2}$$

where M is the money stock, P is the price level and i is the nominal rate of interest. By definition, the real rate of interest is equal to the nominal rate minus the expected rate of inflation, χ i.e.:

$$r = i - \chi \tag{3}$$

Under the natural rate hypothesis, aggregate supply will exceed (fall short of) its natural level, Y*, according to whether the actual inflation rate is above (below) its expected rate. Hence:

$$h\ln\,(Y/Y^*) = \Pi - \chi; \quad h > 0 \tag{4}$$

where Π ($\equiv d\,\ell n(P)/dt$) is the actual inflation rate. Finally, we assume that inflation expectations are formed rationally, as follows:

$$\chi(t) = \underset{s \to \bar{t}}{\mathrm{Lim.}}\, E\{\Pi(t) \mid I(s)\} \tag{5}$$

where I(s) is the information set at time s. The information includes the structure of the model and the values of the state variables at the "beginning" of t, but it does not include the realization of stochastic error terms at t.

Solving (1), (2) and (4) for ln(Y), r and Π (after first using (3) to eliminate i from (2)) results in the following system:

$$\ln(Y) = A + \frac{B}{\beta} \ln\,(M/P) + \chi + \frac{B\sigma}{b} W \tag{6}$$

$$r = \frac{B}{b} \left\{ \frac{b^2\,(A+\alpha)}{B^2} - \frac{1}{\beta} \ln\,(M/P) - \chi - \frac{\sigma}{\beta} W \right\} \tag{7}$$

$$\Pi = h \left\{ A + \frac{B}{\beta} \ln\,(M/P) + (B + 1/h)\,\chi - \ln\,(Y^*) + \frac{B\sigma}{b} W \right\} \tag{8}$$

where:

$$A \equiv \frac{a\beta - \alpha b}{b+\beta} \quad \text{and} \quad B \equiv \frac{b\beta}{b+\beta}$$

Taking the rational expectation of Π in (8) in accordance with definition (5) gives:

$$\chi = -\frac{1}{\beta} \ln (^{M}/_{P}) + \frac{1}{B} \ln (Y^*) - \frac{A}{B} \tag{9}$$

Now use (9) to substitute for χ in (6), (7) and (8) and thus derive:

$$\ln (Y) = \ln (Y^*) + \frac{B\sigma}{b} W \tag{10}$$

$$r = \frac{1}{b} \left\{ a - \ln (Y^*) + \frac{B\sigma}{\beta} W \right\} \tag{11}$$

$$\Pi = -\frac{1}{\beta} \ln (^{M}/_{P}) + \frac{1}{B} \ln (Y^*) - \frac{A}{B} + \frac{hB\sigma}{b} W \tag{12}$$

Equations (10) and (11) illustrate the strict neutrality of monetary policy under the assumption of rational expectations. Both real income and the real rate of interest differ from their natural levels only by a random error; monetary policy can only operate on the rate of inflation.

We now exploit a useful property of the white noise process, $W(t)$. Let $v(t)$ represent a Weiner process with unit variance parameter and incremental variance dt (i.e. $E(dv^2) = dt$). Then, even though the Weiner process is nowhere differentiable, it can be shown that, in the sense of the equivalence of variances, $dv = Wdt$. (See, for example, Arnold (1) ch. 3). Using this together with the definition of Π we may equivalently express (12) as the following stochastic differential equation for the price level:

$$dP = -\frac{P}{\beta} \ln (^{M}/_{P})\, dt + \frac{1}{B} (\ln (Y^*) - A)\, P dt + \frac{hb\sigma}{b} P\, dv \tag{13}$$

Equation (13) is to be regarded as a stochastic differential equation in the sense of Ito and we will now use it to generate a stochastic differential equation in the expected inflation rate. Using Ito"s differentiation formula[1] to differentiate equation (9):

$$d\chi = -\frac{1}{\beta} \frac{dM}{M} + \frac{1}{\beta} \frac{dP}{P} + \frac{1}{B} \frac{dY^*}{Y^*} - \frac{1}{2\beta} \frac{dP^2}{P^2}$$

1. See, for example, Arnold (op. cit.) chapter 5.

Let $^{dM}/_M = udt$, $dY^*/Y^* = y^*dt$ (y^* = constant) and note that $dP^2/P^2 = (hB\sigma/b)^2\, dt + o\,(dt)$. Using this and substituting for dP from equation (13) and χ from equation (9) we derive:

$$d\chi = \frac{1}{\beta}(\chi - u + by^*/B - C^2/2)\,dt + \frac{Cdv}{\beta} \tag{14}$$

where $\quad C \equiv \dfrac{h\beta\sigma}{b+\beta}$

We have chosen to generate an equation in the expected, rather than the actual rate of inflation for mathematical convenience. Nothing substantial hinges on this, however, since the two differ only by a random term with zero mean.

3. THE STOCHASTIC OPTIMIZATION PROBLEM

We have previously noted the fact that the impact of monetary policy will be solely on the rate of inflation. Consequently we take χ as our target (state) variable and u, the proportionate growth rate of the money supply, as our instrument (control) variable. Following Chow (6) we seek a control strategy that will minimise the expected value of a quadratic objective functional, i.e. we wish to:

$$\min_{\{u(t)\}} \; E\int_0^{\infty}\left\{\frac{\gamma}{2}\chi^2 + \frac{\delta}{2}u^2\right\}e^{-rt}\,dt \tag{15}$$

subject to (14) and the assumption that the initial state, $\chi(0)$, is normally distributed with mean $\bar{\chi}_0$ and variance σ_0^2; γ, δ and r are positive constants. The interpretation of the objective functional (15) is that since inflation is regarded as a "bad" we seek to keep it close to zero whilst at the same time not allowing too violent fluctuations in the money stock.

The problem will be solved by the applicatiion of the stochastic version of dynamic programming. (See, for example, Astrom (2)). Suppose that the minimum expected value of (15) if we start at time t in state χ is $e^{-rt}\,V\,(t, \chi)$ where V(.) is continuously differentiable with respect to t and twice continuously differentiable with respect to χ. Then, applying the principle of optimality (see Astrom (op. cit.) for details) we derive:

$$O = \min_{\{u(t)\}} \left\{ \left[\frac{\gamma}{2} \chi^2 + \frac{\delta u^2}{2} \right] e^{-rt} + L\,[e^{-rt} \,.\, V(\chi, t)] \right\} \tag{16}$$

where L[.] is the stochastic differential generator. Applying the differential generator to $e^{-rt}\, V(\chi, t)$ gives:

$$L\,[e^{-rt}\, V(\chi, t)] =$$

$$e^{-rt} \left\{ \frac{\partial V}{\partial t} - rV + \frac{1}{\beta} (\chi - u + \beta y^*/B - C^2/2) \frac{\partial V}{\partial \chi} + \ldots + \frac{C^2}{2\beta^2} \frac{\partial^2 V}{\partial \chi^2} \right\} \tag{17}$$

Substituting (17) into (16), differentiating with respect to u and setting the result equal to zero gives:

$$u(t) = \frac{1}{\delta\beta} \frac{\partial V}{\partial \chi} \tag{18}$$

Substituting this result back into (16) and rearranging we see that $V(\chi, t)$ must satisfy the following non-linear second-order partial differential equation:

$$\frac{C^2}{2\beta^2} \frac{\partial^2 V}{\partial \chi^2} - \frac{1}{2\delta\beta^2} \left(\frac{\partial V}{\partial \chi} \right)^2 + \left(\frac{\chi}{\beta} + \frac{y^*}{B} - \frac{C^2}{2\beta} \right) \frac{\partial V}{\partial \chi} + \frac{\partial V}{\partial t} \tag{19}$$

$$\ldots - rV + \frac{\gamma}{2} \chi^2 = 0$$

The solution of this equation may be found in any textbook on stochastic control theory; it is also explained in Chow (5). Generally, the solution will take the form:

$$V(\chi, t) = \frac{Q}{2} \chi^2 + R\chi + S \tag{20}$$

where Q, R and S are continuous functions of time. Taking the partial derivatives of (20) which appear in (19) and substituting back into (19) suggests the following system of Ricatti differential equations for Q, R and S.

$$\dot{Q} - \frac{1}{\delta\beta^2} Q^2 + \left(\frac{2}{\beta} - r \right) Q + \gamma = 0$$

$$\dot{R} - R\left\{\frac{1}{\delta\beta^2}Q + r - \frac{1}{\beta}\right\} + Q\left\{\frac{y^*}{B} - \frac{C^2}{2\beta}\right\} = 0 \tag{21}$$

$$\dot{S} - rS + \frac{C^2}{2\beta^2}Q + \left(\frac{y^*}{B} - \frac{C^2}{2\beta}\right)R - \frac{1}{2\delta\beta^2}R^2 = 0$$

In general, this system can be solved recursively without much difficulty but for the present problem (infinite right-hand end point and all constant coefficients) it possesses a particularly simple solution. All that is required is to find a (unique) set of constants, Q^*, R^* and S^*, with $Q^*>0$, satisfying system (21); (see, for example, Kushner (10) chapter 11 for details). For the present problem these constants are given by:

$$Q^* = \frac{\delta\beta\{2 - r\beta + [(2-r\beta)^2 + 4\gamma/\delta]^{1/2}\}}{2}$$

$$R^* = \frac{\delta\beta(2\beta y^* - BC^2)\{2 - r\beta + [(2-r\beta)^2 + 4\gamma/\delta]^{1/2}\}}{2B\{r\beta + [(2-r\beta)^2 + 4\gamma/\delta]^{1/2}\}} \tag{22}$$

$$S^* = \frac{1}{2\beta r}\left\{\frac{C^2Q^*}{\beta} + \left(\frac{2\beta y^* - BC^2}{B}\right)R^* - \frac{R^{*2}}{\delta\beta}\right\}$$

Using (18), (20) and (22) we see that it is optimal for the rate of growth of the money supply to follow the linear feedback rule:

$$u(t) = \frac{Q^*\chi(t) + R^*}{\delta\beta} \tag{23}$$

The constant Q^* is of course positive but the sign of R^* depends on the sign of the first bracketed term in the numerator. Clearly, with no uncertainty, $C=o$ and R^* is positive. However, R^* is a decreasing function of the amount of uncertainty, as measured by the variance, σ^2. Consequently, the presence of uncertainty results in a "tighter" optimal money supply policy. It is also of interest to note that the certainty-equivalence principle does not hold in this model; the value of R^* is higher, and thus the optimal feed-back control is different, in the deterministic case.

Let us now examine the behaviour of the inflation rate under this feed-back policy for the money supply. Substituting (23) back into (14) we see that the inflation rate is a well defined stochastic process that evolves according to the following stochastic differential equation:

$$d\chi = -\frac{(Q^* - \delta\beta)}{\delta\beta^2}\,\chi\,dt - \frac{(1 - r\beta)\,R^*}{BQ^*}\,dt + \frac{C}{\beta}\,dv \tag{24}$$

The behaviour of the inflation rate over time may be completely characterised by specifying:

(i) The time – path of the mean (i.e. expected) value of χ, $\bar{\chi}(t)$.

(ii) The time path of the variance, $\Sigma(t)$. In this way we may describe the probability distribution of the inflation rate at each point in time. We cannot, of cource, conceive of a "solution" of (24) in the same way as we can for a deterministic differential equation. The mean value function and the variance function are the solutions of the following deterministic differential equations (see Astrom (op. cit.) chapter 3).

$$\dot{\bar{\chi}} = -\frac{(Q^* - \delta\beta)}{\delta\beta^2}\,\bar{\chi} - \frac{(1 - r\beta)\,R^*}{\beta\,Q^*} \quad ; \bar{\chi}(o) = \bar{\chi}_o$$

$$\dot{\Sigma} = -\frac{2(Q^* - \delta\beta)}{\delta\beta^2}\,\Sigma + \frac{C^2}{\beta^2} \quad ; \Sigma(o) = \sigma_o^2$$

The solutions of these equations are:

$$\bar{\chi}\;(t) = \left\{\bar{\chi}_o + \frac{\delta\beta\,(1 - r\beta)\,R^*}{Q^*\,(Q^* - \delta\beta)}\right\} \exp.\left\{-\frac{(Q^* - \delta\beta)\,t}{\delta\beta^2}\right\} - \frac{\delta\beta\,(1 - r\beta)\,R^*}{Q^*\,(Q^* - \delta\beta)}$$

$$\Sigma(t) = \left\{\sigma_o^2 - \frac{\delta C^2}{2(Q^* - \delta\beta)}\right\} \exp.\left\{\frac{-2\,(Q^* - \delta\beta)\,t}{\delta\beta^2}\right\} + \frac{\delta C^2}{2(Q^* - \delta\beta)} \tag{25}$$

The stability condition is the same as in the deterministic case, that $Q^* > \delta\beta$. In terms of the parameters of the problem this will be satisfied if $r < (\gamma + \delta)/\delta\beta$, i.e. the discount rate must not be too high. For all reasonable values of the parameters we may expect the condition to be met and in what follows we shall assume that it is. Two further quantities of interest are the asymptotic mean and variance of the inflation rate. Provided the stability condition is satisfied, the probability distribution of the inflation rate will eventually converge on the steady state distribution characterised by:

$$\bar{\chi}^* = -\frac{\delta\beta\,(1 - r\beta)\,R^*}{Q^*\,(Q^* - \delta\beta)}$$

$$\Sigma^* = \frac{\delta C^2}{2(Q^* - \delta\beta)}$$

where the "stars" denote asymptotic values. We see that in general the mean inflation rate does not go to zero, but to some finite steady – state value; the sign of this again depends on the sign of R^*; it is thus an increasing function of the amount of uncertainty.

4. CONCLUDING REMARKS

We have shown that the optimal policy for the rate of growth of the money supply is to follow a linear feedback rule. If the stability condition holds, the "slope" coefficient in the feedback rule is greater than unity but the "intercept" is a négative function of the amount of uncertainty in the System. We would emphasise that the model illustrâtes a pure anti - inflation policy since there is no trade-off between output (or unemployment) and inflation in this strict-neutrality-of-money framework. We feel that it may hâve potentially interesting implications for empirical work.

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