A NEW APPROACH ON PSEUDO - ALTERNATING PISTOL MULTIPERMUTATIONS

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ABSTRACT

This paper is dealing with pseudo - alternating pistol multipermutations (PAPM) and the corresponding alternating permutations. Two new methods are described for finding all the PAPM'S of length 2n and the corresponding alternating permutations with the help of permutation trees.

1. INTRODUCTION

Let G [m] the set of the permutations of the set $[m] = \{1, 2, ..., m\}$. Then every permutation of G [m] is denoted by

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & m \\ \sigma(1) & \sigma(2) & \dots & \sigma(m) \end{pmatrix} \equiv \sigma(1)\, \sigma(2) \dots \sigma(m),\ (i, \mu(i)) \in [m]^2$$

The elements of the particular case G[3] are given in (Fig. 1), where every path, the vertices of which belong to 3 levels L_1, L_2, L_3 corresponds to a permutation of G [3] and viceversa.

Moreover using permutation trees, it is always possible to determine a subset of G [m], which satisfies certain properties and/or assumptions.

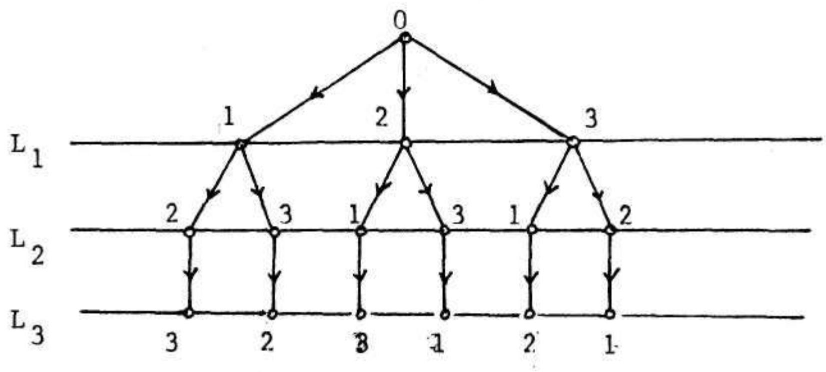

Fig. 1.

Definition 1.1.

A *finite multiset* M of positive integers may be regarded as a collection of positive integers with repetitions allowed such that the total number of elements appearing in the collection is finite. Let $M \equiv \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ where k_i indicates that the element i is repeated k_i times in M, so $\Sigma k_i < \infty$. A multipermutation μ of M is a linear arrangement $\mu(1), \mu(2), \ldots, \mu(\Omega)$ of the elements of M, where $\Omega = \Sigma k_i = [M]$, (4) and will be denoted by $\mu \equiv \mu(1)\, \mu(2) \ldots \mu(m)$, $\mu(i) \in M$. For instance the permutation $\mu = 1123332$ is one of the $7!/2!\ 2!\ 3! = 210$ possible multipermutations of $M = \{1^2, 2^2, 3^3\}$.

Defintion 1.2.

The map $b{:}[m] \rightarrow M$, which is defined by the following conditions

$$b(1) = 1 \quad \text{and} \quad \forall i \geq 1, \quad \begin{aligned} b(2i) &= b(2i-1) \\ b(2i+1) &= b(2i) + 1 \end{aligned}$$

determines a multipermutation called ***basic multipermutation of length m*** (i.e. the multipermutation 1122334455 is a basic multipermutation of length 10).

Definition 1.3.

A pistol multipermutation of length m is a map $\xi{:}[m] \rightarrow M$ with $\xi(i) \leq b(i)$, $\forall\ i \geq 1$, (2) For instance the multipermutation $\xi = 11113342$ is one pistol multipermutation of length 8. We denote P_m the set of pistol of length m.

Definition 1.4.

A pistol multipermutation of length m is said to be a ***pseudo – alternating pistol multipermutation (PAPM)*** of length m if $\forall$ i $\in [m-1]$

$$\zeta(i) \geq \zeta(i+1) \ , \quad \text{if i is odd}$$
$$\zeta(i) \leq \zeta(i+1) \ , \quad \text{if i is even}$$

For instance the multipermutation $\zeta = 11113342$ is a pseudo – alternating pistol multipermutation, displayed in fig. 2.

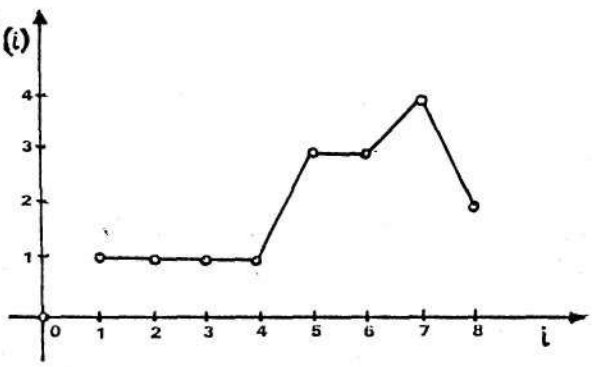

Fig. 2

We denote AP_m the set of PAPM's of length m.

2. AN ALGORITHM FOR FINDING THE SET AP_{2n}.

The Genocchi numbers G_{2n} are integers defined by the relation

$$C_{2n} = \frac{n}{2^{2(n-1)}} T_{2n-1}, \qquad n \geq 1$$

from the Euler numbers T_{2n-1} of second kind, with the exponential generating function

$$\tan t = \sum_{n\geq 1} T_{2n-1} \frac{t^{2n-1}}{(2n-1)!}$$

The values of this sequence for some values of n are given in the Table 1.

TABLE 1

n	1	2	3	4	5	6	7
T_{2n-1}	1	2	16	272	7936	353792	22368256
G_{2n}	1	1	3	17	155	2973	38227

The Genocchi number G_{2n+2} gives the cardinal number of the set AP_{2n} (i.e. $|AP_{2n}| = G_{2n+2}$, $\forall n \geq 1$). For instance $|AP_6| = G_8 = 17$.

In this paper a new method is proposed for finding all the PAPM's. This method uses the parmutation trees (2) to determine the elements of the set AP_{2n} with the help of the following sets:

$$T(j) = \{1, 2, \ldots, j\}$$

$$S(j) = \{j, j+1, j+2, \ldots, n\}$$

$$N(\kappa) = \{\frac{\kappa}{2} + 2, \frac{\kappa}{2} + 3 =, \ldots, n\}$$

where κ is the order of the levels $L_1, L_2, \ldots, L_{2n}$ i.e. L_1 is the first level, L_2 the second level etc.

It must be noted that to the levels $L_1, L_2, \ldots, L_{2n}$ of the tree T, correspond the elements $\zeta(1), \zeta(2), \ldots, \zeta(2n)$ of the PAPM ζ length 2n and $\Gamma_\kappa(j)$ denotes the k-level subset of $[2n]$ with elements the terminal vertices of the arcs with initial vertex j.

The following proposition discribes the algorithm implemented the above method.

Proposition 2.1.

The paths of T which satisfy the following conditions:

(i) $\Gamma_0(0)=1$ and $\Gamma_1(1)=1$

(ii) For every level κ and every vertex j:

$$\Gamma_\kappa(j)=\begin{cases} S(j)-N(\kappa), & \text{if } \kappa=2\rho,\ N(\kappa)\subseteq S(j) \\ T(j), & \text{if } \kappa=2\rho+1 \end{cases}$$

give all the elements of AP_{2n}.

Example 2.2

The following example illustrates the above algorithm:

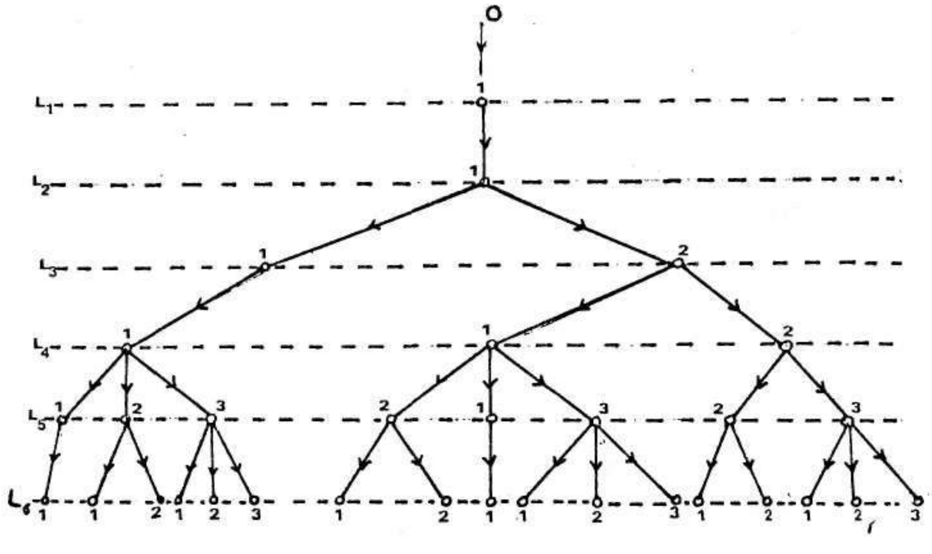

Fig. 3

For every level L_1, L_2, L_3, L_4, L_5, L_6 and every vertex j, we get the following subsets:

L_1: $\Gamma_1(1)=T(1)=\{1\}$

L_2: $\Gamma_2(1)=S(1)-N(2)=\{1, 2, 3\}-\{3\}=\{1, 2\}$

L_3: $\Gamma_3(1) = T(1) = \{1\}$

$\Gamma_3(2) = T(2) = \{1, 2\}$

L_4: $\Gamma_4(1) = S(1) - N(4) = \{1, 2, 3\} - \varnothing = \{1, 2, 3\}$

$\Gamma_4(2) = S(2) - N(4) = \{2, 3\} - \varnothing = \{2, 3\}$

L_5: $\Gamma_5(1) = T(1) = \{1\}$

$\Gamma_5(2) = T(2) = \{1, 2\}$

$\Gamma_5(3) = T(3) = \{1, 2, 3\}$

Therefore $AP_6 = \{111111, 111121, 111122, 111131, 111132, 111133, 112111, 112121, 112122, 112131, 112132, 112133, 112221, 112222, 112231, 112232, 112233\}$.

Example 2.3

For the previous algorithm, a program PASCAL was run for different values of n. For $n = 4$ the output was the following table 2, where the number of PAPM's of length 8 coincides with the Gennocchi number

$$|AP_8| = G_{10} = 155$$

given in the table 1.

TABLE 2

11111111	11111121	11111122	11111131	11111132
11111133	11111141	11111142	11111143	11111144
11112111	11112121	11112122	11112131	11112132
11112133	11112141	11112142	11112143	11112144
11112221	11112222	11112231	11112232	11112233
11112241	11112242	11112243	11112244	11113111
11113121	11113122	11113131	11113132	11113133
11113141	11113142	11113143	11113144	11113221
11113222	11113231	11113232	11113233	11113241
11113242	11113243	11113244	11113331	11113332
11113333	11113341	11113342	11113343	11113344
11211111	11211121	11211122	11211131	11211132
11211133	11211141	11211142	11211143	11211144
11212111	11212121	11212122	11212131	11212132

11212133	11212141	11212142	11212143	11212144
11212221	11212222	11212231	11212232	11212233
11212241	11212242	11212243	11212244	11213111
11213121	11213122	11213131	11213132	11213133
11213141	11213142	11213143	11213144	11213221
11213222	11213231	11213232	11213233	11213241
11213242	11213243	11213244	11213331	11213332
11213333	11213341	11213342	11213343	11213344
11222111	11222121	11222122	11222131	11222132
11222133	11222141	11222142	11222143	11222144
11222221	11222222	11222231	11222232	11222233
11222241	11222242	11222243	11222244	11223111
11223121	11223122	11223131	11223132	11223133
11223141	11223142	11223143	11223144	11223221
11223222	11223231	11223232	11223233	11223241
11223242	11223243	11223244	11223331	11223332
11223333	11223341	11223342	11223343	11223344

3. PAPM'S AND CORRESPONDING ALTERNATING PERMUTATIONS

Definition 3.1.

A permutation $\sigma \equiv (1)\ \sigma(2)\ \ldots\ \sigma(m)$, $m > 2$ is said to be alternating (1) if the terms of its pattern are alternating i.e. $w(\sigma) = rdrd \ldots$ or $W(\sigma) = drdr \ldots$ where $r = \text{rise} = +$, $d = \text{descent} = -$. (i.e. 86572413 is an alternating permutation of length 8.

To every PAPM of length 2n corresponds one alternating permutation of length $2n+1$ (2) which can be found with the help of the following proposed procedure:

Proposition 3.2

i) For every PAPM ζ length 2n we define the map ζ': $[2n] \to [2n+1]$ by the following conditions:

$$\zeta'(1) = 1$$

$$\left.\begin{array}{l} \zeta'(2i) = i + 1 - \zeta\,(2i-1) \\ \zeta'(2i+1) = i + 2 - \zeta\ \ (2i) \end{array}\right\} \text{ for every } i : 2 \leq 2i \leq n$$

ii) For every PAPM $\zeta' \in A_{2n+1}$, we associate the map ζ'': $[2n+1] \rightarrow [0, 2n]$, by the following condition:

$$\forall i \in [2n+1], \qquad \zeta''(i) = 2(\zeta'(i) - 1)$$

iii) Finally the elements of the unique corresponding alterating permutation $\sigma = \sigma(1)\,\sigma(2)\,\ldots\,\sigma(n)$ is given by the relation

$$\sigma(i) = (\zeta''(i)+1)^{\text{th}} \quad \text{element of the set } A_i,$$
$$i = n,\ n-1,\ n-2,\ \ldots,\ 2,\ 1$$

where $A_{n,<} = [n]$
and $A_{n-1,<} = A_{n,<} - \{\sigma(i)\}$

Example 3.3

For the corresponding alternation permutation to the PAPM $\zeta = 11113342$.

i) For $\zeta = 11113342$, $\zeta \in A_8$ we take $\zeta' = 112231214$, $\zeta' \in A_9$.

ii) For $\zeta' = 112231214$, $\zeta \in A_9$ we take $\zeta'' = 002240206$.

iii) Finally, for $\zeta'' = 002240206$, we have:

$A_9 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
$\sigma(9) = \zeta'(9) + 1 = 7$ element of A_9 i.e. 7, $\sigma(9) = 7$.

$A_8 = \{1, 2, 3, 4, 5, 6, 8, 9\}$
$\sigma(8) = \zeta''(8) + 1 = 1$ element of A_8 i.e. 1, $\sigma(8) = 1$.

$A_7 = \{2, 3, 4, 5, 6, 8, 9\}$
$\sigma(7) = \zeta''(7) + 1 = 3$ element of A_7 i.e. 4, $\sigma(7) = 4$

$A_6 = \{2, 3, 5, 6, 8, 9\}$
$\sigma(6) = \zeta''(6) + 1 = 1$ element of A_6 i.e.2, $\sigma(6) = 2$

$A_5 = \{3, 5, 6, 8, 9\}$
$\sigma(5) = (\zeta''(5) + 1) = 5$ element of A_5 i.e. 9, $\sigma(5) = 9$

$A_4 = \{3, 5, ,6 ,8\}$
$\sigma(4) = \zeta''(4) + 1 = 3$ element of A_4 i.e. 6, $\sigma(4) = 6$

$A_3 = \{3, 5, 8\}$

$\sigma(3) = \zeta''(3) + 1 = 3$ element of A_3 i.e. 8, $\sigma(3) = 8$

$A_2 = \{3, 5\}$

$\sigma(2) = \zeta''(2) + 1 = 1$ element of A_2 i.e. 3, $\sigma(2) = 3$

$A_1 = \{5\}$

$\sigma(1) = \zeta''(1) + 1 = 1$ element of A_1 i.e. 5, $\sigma(1) = 5$

Hence, the corresponding alternating permutation is $\sigma = 538692417$.

Example 3.4

For the above method a program PASCAL was run for different values of n and all the PAPM's of the table 1 have been found. For $n = 4$ (i.e. of length 8) the output is given in table 3.

TABLE 3

	ζ	ζ'	ζ''	σ
1)	11111111	112233445	002244668	214365879
2)	11111121	112233435	002244648	214376859
3)	11111122	112233434	002244646	214386957
4)	11111131	112233425	002244628	215476839
5)	11111132	112233424	002244626	215486937
6)	11111133	112233423	002244624	216487935
7)	11111141	112233415	002244608	325476819
8)	11111142	112233414	002244606	325486917
9)	11111143	112233413	002244604	326487915
10)	11111144	112233412	002244602	426587913
11)	11112111	112232445	002242668	215463879
12)	11112121	112232435	002242648	216473859
13)	11112122	112232434	002242646	216483957
14)	11112131	112232425	002242628	216574839
15)	11112132	112232424	002242626	216584937
16)	11112133	112232423	002242624	217684935
17)	11112141	112232415	002242608	326574819
18)	11112142	112232414	002242606	326584917
19)	11112143	112232413	002242604	327684915
20)	11112144	112232412	002242602	427685913
21)	11112221	112232335	002242448	217483659
22)	11112222	112232334	002242446	218493657

23)	11112231	112232325	002242428	217584639
24)	11112232	112232324	02242426	218594637
25)	11112233	112232323	002242424	218694735
26)	11112241	112232315	002242408	327584619
27)	11112242	112232314	002242406	32859715
28)	11112243	112232313	002242404	328694617
29)	11112244	112232312	002242402	428695713
30)	11113111	112231445	002240668	325461879
31)	11113121	112231435	002240648	326471859
32)	11113122	112231434	002240646	326481957
33)	11113131	112231425	002240628	426571839
34)	11113132	112231424	002240626	426581937
35)	11113133	112231423	002240624	427681935
36)	11113141	112231415	002240608	436572819
37)	11113142	112231414	002240606	436582917
38)	11113143	112231413	002240604	437682915
39)	11113144	112231412	002240602	547682913
40)	11113221	112231335	002240448	327481659
41)	11113222	112231334	002240446	328491657
42)	11113231	112231325	002240428	427581639
43)	11113232	112231324	002240426	428591637
44)	11113233	112231323	002240424	428691735
45)	11113241	112231315	002240408	437582619
46)	11113242	112231314	002240406	438592617
47)	11113243	112231313	002240404	438692715
48)	11113244	112231312	002240402	548692713
49)	11113331	112231225	002240228	527681439
50)	11113332	112231224	002240226	528691437
51)	11113333	112231223	002240224	628791435
52)	11113341	112231215	002240208	537682419
53)	11113342	112231214	002240206	538692417
54)	11113343	112231213	002240204	638792415
55)	11113344	112231212	002240202	648792513
56)	11211111	112133445	002044668	324165879
57)	11211121	112133435	002044648	324176859
58)	11211122	112133434	002044646	324186957
59)	11211131	112133425	002044628	425176839
60)	11211132	112133424	002044626	425186937
61)	11211133	112133423	002044624	426187935
62)	11211141	112133415	002044608	435276819
63)	11211142	112133414	002044606	435286917
64)	11211143	112133413	002044604	436287915
65)	11211144	112133412	002044602	546287913
66)	11212111	112132445	002042668	425163879
67)	11212121	112132435	002042648	426173859
68)	11212122	112132434	002042646	426183957
69)	11212131	112132425	002042628	526174839
70)	11212132	112132424	002042626	526184937

71)	11212133	112132423	002042624	627184935
72)	11212141	112132415	002042608	536274819
73)	11212142	112132414	002042606	536284917
74)	11212143	112132413	002042604	637284915
75)	11212144	112132412	002042602	647285913
76)	11212221	112132335	002042448	427183659
77)	11212222	112132334	002042446	428193657
78)	11212231	112132325	002042428	527184639
79)	11212232	112132324	002042426	528194637
80)	11212233	112132323	002042424	628194735
81)	11212241	112132315	002042408	537284619
82)	11212242	112132314	002042406	538294617
83)	11212243	112132313	002042404	638294715
84)	11212244	112132312	002042402	648295713
85)	11213111	112131445	002040668	435261879
86)	11213121	112131435	002040648	436271859
87)	11213122	112131434	002040646	436281957
88)	11213131	112131425	002040628	546271839
89)	11213132	112131424	002040626	546281937
90)	11213133	112131423	002040624	647281935
91)	11213141	112131415	002040608	546372819
92)	11213142	112131414	002040606	546382917
93)	11213143	112131413	002040604	647382915
94)	11213144	112131412	002040602	657482913
95)	11213221	112131335	002040448	437281659
96)	11213222	112131334	002040446	438291657
97)	11213231	112131325	002040428	547281639
98)	11213232	112131324	002040426	548291637
99)	11213233	112131323	002040424	648291735
100)	11213241	112131315	002040408	547382619
101)	11213242	112131314	002040406	548392617
102)	11213243	112131313	002040404	648392715
103)	11213244	112131312	002040402	658492713
104)	11213331	112131225	002040228	657281439
105)	11213332	112131224	002040226	658291437
106)	11213333	112131223	002040224	768291435
107)	11213341	112131215	002040208	657382419
108)	11213342	112131214	002040206	658392417
109)	11213343	112131213	002040204	768392415
110)	11213344	112131212	002040202	768492513
111)	11222111	112122445	002022668	526143879
112)	11222121	112122435	002022648	627143859
113)	11222122	112122434	002022646	628143957
114)	11222131	112122425	002022628	627154839
115)	11222132	112122424	002022626	628154937
116)	11222133	112122423	002022624	728164935
117)	11222141	112122415	002022608	637254819
118)	11222142	112122414	002022606	638254917

119)	11222143	112122413	002022604	738264915
120)	11222144	112122412	002022602	748265913
121)	11222221	112122335	002022448	728143659
122)	11222222	112122334	002022446	829143657
123)	11222231	112122325	002022428	728154639
124)	11222232	112122324	002022426	829154637
125)	11222233	112122323	002022424	829164735
126)	11222241	112122315	002022408	738254619
127)	11222242	112122314	002022406	839254617
128)	11222243	112122313	002022404	839264715
129)	1122244	112122312	002022402	849265713
130)	11223111	112121445	002020668	536241879
131)	11223121	112121435	002020648	637241859
132)	11223122	112121434	002020646	638241957
133)	11223131	112121425	002020628	647251839
134)	11223132	112121424	002020626	648251937
135)	11223133	112121423	002020624	748261935
136)	11223141	112121415	002020608	647352819
137)	11223142	112121414	002020606	648352917
138)	11223143	112121413	002020604	748362915
139)	11223144	112121412	002020602	758462913
140)	11223221	112121335	002020448	738241659
141)	11223222	112121334	002020446	839241657
142)	11223231	112121325	002020428	748251639
143)	11223232	112121324	002020426	849251637
144)	11223233	112121323	002020424	849261735
145)	11223241	112121315	002020408	748352619
146)	11223242	112121314	002020406	849352617
147)	11223243	112121313	002020404	849362715
148)	11223244	112121312	002020402	859462713
149)	11223331	112121225	002020228	758261439
150)	11223332	112121224	002020226	859261437
151)	11223333	112121223	002020224	869271435
152)	11223341	112121215	002020208	758362419
153)	11223342	112121214	002020206	859362417
154)	11223343	112121213	002020204	869372415
155)	11223344	112121212	002020202	869472513

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