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ON THE SOLUTION OF SADDLEPOINT PROBLEMS IN CONTINUOUS TIME RATIONAL EXPECTATIONS MODELS*

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Abstract

A generalised solution method for rational expectations models which are representable by systems of first-order linear differential equations with constant coefficients is introduced. The existing literature relies on the existence of a similarity transformation which can be used to diagonalize the coefficient matrix of the model. The method presented in this paper may also be used when such a transformation does not exist. A simple example for which a diagonalizing transformation does not exist is used to illustrate the application of the solution method.

1. Introduction

The purpose of this paper is to introduce a generalized method of solution for those rational expectations models which are representable by systems of linear differential equations with constant coefficients. These models are characterised by a generalised saddle - point property and the solution, as well as describing dynamic behaviour along the saddle-point path, also determines the restricitions (usually boundary conditions) which are necessary to render the solution stable[1]. Typically the non-predetermined, forward looking, variables exhibit a discrete "jump" so as to place the solution on the stable manifold. In this paper we confine our attention to those cases where the number of pre - determined, backward looking, variables is equal to the number of stable roots of the characteristic equation of the model. Howeer, cases where there are more stable roots than pre - determined variables should also be straightforward to handle in many cases if suitable restrictions can be formulated.

The solution method adopted in much of the literature to date uses a similarity transformation to diagonalize the coefficient matrix of the model. (See, for example, Buiter (2) and Dixit (3)). The method presented here also deals successfully with those cases where such a transformation does not exist.

* I am grateful to Tony Sampson for helpful discussions, but the usual caveat applies.

In section 2 we briefly review the method of Buiter and Dixit, in section 3 we describe the generalised method and in Section 4 we give a simple economic example for which a diagonalizing transformation does not exist. Finally, in section 5, we offer some concluding remarks.

2. The case of a diagonalizable coefficient matrix

Consider the following linear dynamic model:

$$\dot{X} = A(X - \bar{X}) \tag{1}$$

where X is an n-vector, $\bar{X}$ is the n-vector of equilibrium values of the components of X and A is an nxn matrix of constants. For simplicity we assume here that $\bar{X}$ is constant; i.e. that all shocks are current and permanent. However, following the work of Buiter (op. cit.) the case where $\bar{X}$ is time-varying is straightforward enough.

Suppose that λ is a characteristic root of A and that λ has multiplicity k, for $k \geqslant 1$. Suppose also that, corresponding to λ, we can find a set of k linearly independent left characteristic vectors of A. If this holds for every distinct characteristic root of A then there exists a similarity transformation that diagonalizes A. Note that these conditions, which are necessary and sufficient for A to be diagonalizable, are rather less restrictive that the often-quoted condition that the n characteristic roots of A should be distinct. This latter condition is of course sufficient, but not necessary.

Suppose that A satisfies these conditions and let M be the non-singular nxn matrix whose rows are the linearly independent left characteristic vectors of A. Now let:

$$Y = M(X - \bar{X}) \tag{2}$$

so that:

$$X = M^{-1}Y + \bar{X} \tag{3}$$

and

$$\dot{X} = M^{-1}\dot{Y} \tag{4}$$

Substituting (3) and (4) into (1) and rearranging gives:

$$\dot{Y} = MAM^{-1}Y = \Delta Y \tag{5}$$

Where Δ is the nxn diagonal matrix whose diagonal entries are the characteristic roots of A. The solution of (5) is:

$$Y(t) = \exp.(t\Delta)\, Y(o) \tag{6}$$

where exp. $(t\Delta)$ is the diagonal matrix whose diagonal entries are exp. $(\lambda_i t)$, where the λ_i's are the diagonal entries of Δ i.e. the characteristic roots of A. Thus, substituting from (2) into (6) and rearranging,

$$X(t) = \bar{X} + M^{-1} \exp.(t\Delta)\, M\, (X(o) - \bar{X}) \tag{7}$$

Finally, we derive conditions under which the model converges to the equilibrium vector, $\bar{X}$. Assume that the first n_1 elements of X are pre-determined variables and the last $n - n_1$ are non pre-determined, "forward looking", variables. Assume also that A has $n - n_1$ characteristic roots with non-negative real parts and that these are associated with the last $n - n_1$ rows Δ (and thus exp $(t\Delta)$). Denote the n_1 and $n - n_1$ sub vectors of X(o) and $\bar{X}$ by $X_1(o)$, $X_2(o)$, $\bar{X}_1$ and $\bar{X}_2$ respectively. Then in order to neutralize the effects of the unstable roots we need to solve the following equation for the initial values of the non pre-determined variables, $X_2(o)$.

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x_1(o) - \bar{x}_1 \\ x_2(o) - \bar{x}_2 \end{bmatrix} \begin{bmatrix} M_{11}(x_1(o) - \bar{x}_1) + M_{12}(x_2(o) - \bar{x}_2) \\ \underline{O} \end{bmatrix}$$

where $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = M$ is partitioned conformably.

ie., providing M_{22} is non-singular,

$$X_2(o) = \bar{X}_2 - M_{22}^{-1} M_{21} (X_1(o) - \bar{X}_1) \tag{8}$$

These results are, essentially, the results derived by Dixit (3) and also reported by Buiter (op. cit.).

3. A more general solution method

In this section we relax the assumption, made by Buiter and Dixit, that A is diagonalizable. We introduce a rather more general similarity transformation, which reduces A to a Jordan Matrix, and which may be used irrespective of whether A is diagonalizable[2].

It can be shown that for *any* square matrix A there exists a square non-singular matrix V such that:

$$VAV^{-1} = J = \begin{bmatrix} \lambda_1{*} & & \underline{O} \\ & \lambda_2{*} & \\ \underline{O} & & \lambda_n{*} \end{bmatrix} \tag{9}$$

where the elements on the diagonal are the characteristic roots of A and the elements, *, on the super-diagonal may be either 0 or 1; all other elements are zero[3]. It is helpful to consider J (which is known as a Jordan matrix) as the direct sum[4] of a number of Jordan blocks each of the form:

$$J_i = \begin{bmatrix} \lambda_i 1 & & \underline{O} \\ & \lambda_i 1 & \\ & & 1 \\ \underline{O} & & \lambda_I \end{bmatrix} \tag{10}$$

where λ_i is one of the characteristic roots of A and all elements on the super-diagonal are unity. Suppose λ_i is of multiplicity k and that there are h linearly independent left characteristic vector corresponding to λ_i ($h \leqslant k$). Then associated with λ_i there are h such Jordan blocks and these blocks do not increase in size going from left to right along the diagonal of J. In cases where $k-h$ is "large" and $h > 1$ there may be some difficulty in determining the size of each J_i. However, these cases are likely to be rare in practice and we shall not dwell on them here. The interested reader is referred to Finkbeiner (4) for further details.

To construct the matrix V we proceed as follows. For each set of blocks corresponding to a particular characteristic root we have a set of h linearly independant characteristic vectors. Suppose the first few blocks of J are of order pxp, qxq, rxr, and so on. Then the p'th, $(p+q)$' th, $(p+q+r)$'th rows of V are these characteristic vectors. It does not matter which vector is assigned to which block, so long as it is associated with the characteristic root which appears on the diagonal of that block. To find the remaining rows of V write the transformation, (9), in partitioned form as:

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \end{bmatrix} \tag{11}$$

The last row in each sub-matrix V_i is the characteristic vector corresponding to the diagonal element of J_i. To find the remaining rows of V_i note that

$$[V_i]\ [A] = [J_i]\ [V_i]$$

Suppose J_i is of order rxr. Then V_i is rxn and multiplying out gives the following equations (where V_{ij} denotes the j'th row of matrix V_i):

$$V_{ij} A = \lambda_i V_{ij} + V_{i,\,j+1} \;\; ; \; j = 1, \ldots, r-1$$

These may be solved backwards, recursively, since the elements of V_{ir} are known. (These rows are known as generalised characteristic vectors for A – see Noble and Daniel (6)). Note that if A satisfies the conditions for diagonalization given in the previous section then $J = \Delta$ and $V = M$. Hence the results of Dixit and Buiter are a special case of this more general procedure.

Using this transformation, the solution to (1), referring to equations (2) to (7) is:

$$X(t) = \bar{X} + V^{-1} \exp(tJ)\ V\ (X(0) - \bar{X}) \tag{12}$$

Since J is block diagonal so is exp(tJ) and a typical pxp block is:

$$\exp(tJ_i) = \begin{bmatrix} 1 & t & t^2/2! - - - - & t^{p-1}/(p-1)! \\ 0 & 1 & t - - - - - - & t^{p-2}/(p-2)! \\ 0 & 0 & 1 & \\ & & & \\ 0 & 0 & 0 \qquad\qquad 0 & 1 \end{bmatrix} \exp(\lambda_i t) \tag{13}$$

Clearly, the conditions under which the model converges are that the initial conditions for the non pre-determined variables satisfy:

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} X_1(o) - \bar{X}_1 \\ X_2(o) - \bar{X}_1 \end{bmatrix} = \begin{bmatrix} V_{11}(X_1(o) - \bar{X}_1) + V_{12}(X_2(o) - \bar{X}_2) \\ \underline{O} \end{bmatrix}$$

ie. providing V_{22} is not singular

$$X_2(o) = \bar{X}_2 - V_{22}^{-1} V_{21}(X_1(o) - \bar{X}_1) \quad (14)$$

4. A simple economic example

In a recent paper Giavazzi and Wyplosz (5) discuss the nature of the solution of the model considered here when some of the characteristic roots of A are zero[5]. They demonstrate that zero roots are a real possibility in such models and showed that a consequence of zero roots was that the model would exhibit hysteresis. Here we adapt the example used by Giavazzi and Wyplosz to illustrate the solution procedure described in section 3 above.

Consider the following simple IS-LM model of a closed economy under rational expectations.

$$m - p = ay - br \quad (15)$$

$$r = i + \dot{p} \quad (16)$$

$$y = \delta - \beta i \quad (17)$$

Equation (15) is the LM equation; m, p and y are, respectively, the logarithms of the money stock, price level and real output and r is the nominal interest rate. Equation (16) is the Fischer equation with i the real rate of interest; (17) is the IS equation. Suppose that monetary policy is used to control the nominal interest rate. Substituting (17) into (16) gives

$$\dot{p} = r + (y - \delta)/\beta \quad (18)$$

To close the model we must specify the dynamic behaviour of r and y. Following Giavazzi and Wyplosz we make the simplest possible assumptions:

$$\dot{y} = \gamma(y^* - y) \quad (19)$$

where y^* is the (constant) "Natural" level of real output and γ is a positive constant. Finally, if r is constant, then:

$$\dot{r} = 0 \tag{20}$$

Equations (18), (19) and (20) constitute the model which may be expressed as:

$$\dot{x} = A(x - \overline{x}) \tag{21}$$

where $x = [y,\ p,\ r]^T$, $\overline{x} = [y^*,\ p^*,\ (\delta - y^*)/\beta]^T$

(p^* is *any* arbitrary price level: It is well known that such models have an indeterminate price level);

$$A = \begin{bmatrix} -\gamma & 0 & 0 \\ 1/\beta & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now A *cannot* be diagonalized. Its characteristic roots are $-\gamma$ for which [1, 0, 0] is a left characteristic vector and zero (of multiplicity 2) for which the *only* left characteristic vector is (a scalar multiple of) [0, 0, 1].

Following the procedure of section 3 we find that a generalised characteristic vector for the zero root is $[1/\beta\gamma,\ 1,\ 0]$. Hence:

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 1/\beta\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{and } VAV^{-1} = J = \left[\begin{array}{c|cc} -\gamma & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

where the partitioning shows the two Jordan Blocks of which J is composed. (12) and (14) may now be applied in order to write out the complete solution.

5. Concluding remarks

Just how important the problem of a non-diagonalizable coefficient matrix is in practice is probably an empirical matter. However, we have shown that it is relatively easy to construct a theoretical model with this property without making any extreme assumptions. The solution method introduced in the paper provides a general method of dealing with such cases.

An alternative solution procedure would be to tackle the problem directly using the Laplace transform. Using the complex inversion formula, the stability conditions could be derived by setting the residues corresponding to the unstable roots (poles) equal to zero[6]. The methods presented in this paper, however, are probably easier to apply.

Footnotes

1. More accurately, for the solution to converge to a finite steady state.

2. Blanchard and Kahn (1) in a paper concerned with the solution of *difference* equation models *mention* the Jordan matrix. However they do not describe the construction of the transformation matrix and the only example given has a diagonalizable coefficient matrix. The *significance* of this reference certainly does not seem to have been grasped.

3. See, for example, Finkbeiner (4), chapter 7.

4. The direct sum of two square matrices is defined as follows. Let A be nxn and let B be mxm. Then:

$$A \oplus B = \begin{bmatrix} A & \underline{O} \\ \underline{O} & B \end{bmatrix}$$

is of order $(m+n) \times (m+n)$.

5. The solution given by Giavazzi and Wyplosz also assumes that the coefficient matrix is diagonalizable.

6. See, for example, Queen (7), for details of this method.

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