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# DISCRETE - CONTINUOUS TIME OPTIMAL GROWTH MODELS*

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Abstract

Optimal growth models have been either of discrete - or continuous - time formulation. This paps breaks away from this tradition. It considers a model in which consumption takes place at discrete time: while production is continuous, and a model with consumption at all points in time, while output : produced at discrete times. Such formulations make sense and are realistic. First order conditions ar obtained and it is shown that they characterize unique globally optimal solutions. These conditions ar interpreted and their relation to the Euler equation is explained. The discussion reveals a number c interesting aspects of the two models.

## 1. Introduction

Optimal growth theory took off with the original contribution by Ramsey am flourished in the sixties and early seventies[1]. It attracted contributions from . number of mathematical economists and economic theorists and applications hav continued in a range of areas. The results obtained have enhanced our understandin, of the principles which govern the optimal allocation of resources over time. *J-* typical model consists of the maximization of a welfare criterion which depends 01 consumption over time subject to Technological and resource constraints over th same period. The issue is the planning of the optimal intertemporal allocation o resources. A basic result that emerges in the dynamic equation which connects th decisions from one point in time to the next is the equality between the margina rate of substitution in consumption and the marginal rate of transformation ii production. Hence in planning the intertemporal allocation of resources the rul which holds instantaneously in the static model must now hold at all points in time

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Two types of models have been considered in the literature. These are the discrete - time formulation and the continuous - time formulation. In the discrete - time formulation the welfare function is the sum of discounted utilities of consumption and, at equally spaced intervals, the production constraint relates consumption and capital stock decisions. In the continuous - time formulation the welfare functional is the integral of discounted instantaneous utilities of consumption and the production constraint relates at each point in time consumption and investment decisions. In both types of models the choice is between more consumption today and more consumption tomorrow and optimal planning implies that the consumption and capital stock paths solve the dynamical equation which describes the motion of the system when the marginal conditions for optimality are satisfied. In analysing the above models the economists have made a number of mathematical contributions and have introduced devices to tackle in particular the issue of convergence of the welfare criterion. However, in spite of these mathematical adjustments, the fact remains that both the discrete - and continuous - time models are usually cast in such a form so that they become easily tractable mathematical problems with the mathematics required being readily available.

The discrete - time models are analysed through the application of rules of ordinary differential calculus and the continuous - time models through those of the calculus of variations and its extensions, dynamic programming and optimal control theory. The first order conditions for dynamic optimization assume a natural interpretation. A relatively easy argument employing the strict concavity of the utility function proves that the paths of the first order conditions are the unique globally optimal paths.

Most optimal growth models in the literature are in continuous - time form. This is not due to the fact that such formulations are deemed to be more realistic but mainly because the presentation of the results is much neater in continuous time, especially in the case of stochastic models. In general the continuity or piecewise continuity of the functions is simpler to visualize and the asymptotic tendency of the paths easier to depict. On the other hand, economic interpretations are easier in discrete - time models. Even in continuous - time models, in order to obtain the economic interpretation of the Euler - Lagrange equation, one assumes, as a first approximation, a discrete - time formulation with finite but small time - increments and obtains through ordinary calculus a relation which is easy to interpret. The Euler - Lagrange equation emerges in the limit and assumes a natural interpretation. Both types of models are very useful.

This paper breaks away from the above tradition. It starts with the observation that there are economic problems of optimal allocation of resources which do not fall naturally into either of the above categories. It models such situations, and obtains the mathematical characterization of the optimal paths and the economic interpretation of the optimality conditions.

In the models formulated here consumption decisions are not synchronized with the production process. Such formulations make sense and are realistic. On the other hand this implies that the optimal allocation of resources over time must now take place through discrete - continuous time decisions. This implies, in terms of mathematics, that techniques from both ordinary calculus and the calculus of variations must now be employed.

In the first of our models consumption takes place at discrete times while production is continuous (Model 1). For example the individual visits the market once a "week", while money in the bank grows at a continuous compound rate of interest. Utility each week is a function of the quantities available for consumption in that week. As we shall see, the maximization problem is that of choosing optimally a finite number of variables.

Model 2 could describe the following situation. Wheat planted now produces wheat next period, but wheat retained for consumption is spread over the whole interval with the instantaneous utility being a funstion of the consumption rate. Mathematically there are in effect two problems. One is to allocate, optimally, fixed quantities over given intervals and the other is to choose optimally the capital stock at discrete times.

In both models the utility and production functions are strictly concave and future utilities are discounted. A zero discount rate would have simplified the calculations.

For both models we obtain first order conditions for optimality and employ the strict concavity of the functions to show that these conditions characterize a unique globally optimal solution.

Now as explained above in both discrete-and continuous - time models an interior solution requires that the marginal rate of substitution in consumption be equal to the marginal rate of transformation in production, between adjacent periods. In continuous models adjacent periods are t and t + dt, where dt is taken to be very small.

For both Model 1 and 2 here, we obtain analogous interpretations of the first order conditions. In particular in Model 2 we have equality between marginal rate of substitution and marginal rate of transformation within and between periods.

Finally we explain how by allowing the intervals to become shorter and shorter we obtain in both cases, the familiar Eulerian relation of the calculus of variations.

## 2. Consumption at Discrete Times and Production Continuous in Time

We now consider Model 1

Maximise $U = \sum_{i=0}^{n} \delta^i \, u(c_i)$

Subject to

$$c_o + k_o^+ = W$$

$$c_1 + k_1^+ = \int_o^1 + f(k(t))\, dt + k_o^+ \qquad (1)$$

$$c_2 + k_2^+ = \int_1^2 f(k(t))\, dt + k_1^+$$

....................................

$$c_n = \int_{n-1}^{n} f(k(t))\, dt + k_{n-1}^+$$

$$\frac{dk(t)}{dt} = f(k(t)) \quad \text{for } t\varepsilon[i,\, i+1], \quad \text{initial condition } k_i^+,\ c_i,\ k_i^+ \geqslant 0$$

where W is a positive constant and $\delta = \frac{1}{1+\beta}$ with $\beta > 0$. All intervals are of unit length, c denotes consumption and k capital. The same good can be used both for consumption and production purposes. At time $t = i$ a quantity $k_i^+$ of the good is allocated to production and since $\frac{dk(t)}{dt} = f(k(t))$ the total quantity available at $t = i+1$ is $k_{i+1}^- = \int_i^{i+1} f(k(t))\, dt + k_i^+$. This is now divided between consumption and capital input in the new period, i.e. $k_{i+1}^+ + c_{i+1} = k_{i+1}^-$. The process starts with an initial quantity of the good W.

We make the following assumptions. For the instantaneous production function we assume $f(k) > 0$, $f'(k) > 0$ and $f''(k) < 0$, and for the utility function $u'(c) > 0$, $u''(c) < 0$ and $u(0) = \infty$. A positive terminal capital stock requirement could have been accommodated easily. Also a free disposal assumption would not have made any difference to the solution since by assumption $u'(c) > 0$.

Now in order to obtain the solution we argue as follows. Within the intervals there is nothing to choose. Given $k_i^+$, the total quantity of the good, available at time $i+1$ is $\int_i^{i+1} f(k(t))dt + k_i^+$.

We form the Lagrangian function

$$V = \sum_{i=0}^{n} \delta^i u(c_i) + \lambda_o(W - c_o - k_o^+) + \lambda_1\left(\int_0^1 f(k(t))\,dt + k_o^+ - c_1 - k_1^+\right) + \lambda_2\left(\int_1^2 f(k(t))\,dt + \right.$$

$$\ldots\ldots$$

$$\left. k_1^+ - c_2 - k_2^+\right) + \ldots \lambda_{n-1}\left(\int_{n-2}^{n-1} f(k(t))\,dt + k_{n-1}^+\ c_{n-1} - k_{n-1}^+\right) + \lambda_n\left(\int_{n-1}^{n} f(k(t))\right.$$

$$\left. dt + k_{n-1}^+ - c_n\right) \qquad (2)$$

First order conditions for a maximum are

$$\frac{\partial V}{\partial c_o} = u'(c_o) - \lambda_o = 0 \qquad \frac{\partial V}{\partial k_o^+} = -\lambda_o + \lambda_1\left(\int_0^1 f'(k)\,\frac{\partial k\,(t, k_o^+)}{\partial k_o^+}\,dt + 1\right) = 0$$

$$\frac{\partial V}{\partial c_1} = \delta u'(c_1) - \lambda_1 = 0 \qquad \frac{\partial V}{\partial k_1^+} = -\lambda_1 + \lambda_2\left(\int_1^2 f'(k)\,\frac{\partial k\,(t, k_1^+)}{\partial k_1^+}\,dt + 1\right) = 0 \qquad (3)$$

$$\ldots\ldots$$

$$\frac{\partial V}{\partial c_n} = \delta^n u'(c_n) - \lambda_n = 0 \qquad \frac{\partial V}{\partial k_{n-1}^+} = -\lambda_{n-1} + \lambda_n\left(\int_{n-1}^{n} f'(k)\,\frac{\partial k\,(t, k_{n-1}^+)}{\partial k_{n-1}^+}\,dt + 1\right) = 0$$

$$W - c_o - k_o^+ = 0,\ \int_o^1 f(k(t))dt + k_o^+ - c_1 - k_1^+ = 0, \ldots,\ \int_{n-1}^{n} f(k(t))dt + k_{n-1}^+ - c_n = 0$$

Conditions (3) and $\frac{dk}{dt} = f(k)$ in $[i, i+1]$ characterize a unique, globally optimal vector $(c_o^*, c_1^*, \ldots, c_n^*, k_o^{+*}, k_1^{+*}, \ldots, k_{n-1}^{+*})$. In order to obtain this result, first we establish that $z_i = \int_i^{i+1} f(k(t))dt$ is strictly concave in $k_i^+$.

We rewrite $\frac{dk}{dt} = f(k)$ as $\frac{dt}{dk} = \frac{1}{f(k)}$ which implies $t = \int_{k_i}^{k} \frac{dk'}{f(k')} + i = [\varnothing(k')]_{k_i^+}^{k} + i$. Hence $\varnothing(k) = t + \varnothing(k_i^+) - i$ and $k = h(t + \varnothing(k_i^+) - i)$ where $h = \varnothing^{-1}$, and $\varnothing'(k) = \frac{1}{f(k)}$. It follows that $\varnothing'(k)\ \frac{\partial k}{\partial k_i^+} = \varnothing'(k_i^+)$ and $\frac{\partial k}{\partial k_i^+} = \frac{f(k)}{f(k_i^+)} > 0$. We also have $\frac{\partial^2 k}{\partial^2 k_i^+} = \frac{f(k)}{f(k_i^+)^2}\,(f'(k) - f'(k_i^+)) < 0$, for $t > t_i$, because of the strict concavity of $f(k)$ and the fact that $k(t) > k_i^+$. It follows that $\frac{\partial z_i}{\partial k_i^+} = \int_i^{i+1} f'(k(t))\,\frac{\partial k}{\partial k_i^+}\,dt > 0$ and $\frac{\partial^2 z_i}{\partial^2 k_i^+} = \int_i^{i+1}\left(f''(k(t))\left(\frac{\partial k}{\partial k_i^+}\right)^2 + f'(k(t))\,\frac{\partial^2 k}{\partial^2 k_i^+}\right)dt < 0$. Hence $z_i$ is strictly concave in $k_i^+$.

We now turn to the global optimality argument.

Proposition. If $(c_o^*, c_1^*, \ldots, c_n^*, k_o^{+*}, k_1^{+*}, \ldots, k_{n-1}^{+*})$ satisfies (3), for some $(\lambda_o, \lambda_1, \ldots, \lambda_n)$, then it is the unique globally optimal consumption and capital stock vector.

Proof. Obviously all λ'l will be positive. Consider any other feasible consumption and capital stock vector such that at least one $c_i \neq c_i^*$. The strict concavity of u implies

$$\sum_{i=o}^{n} \delta^i u(c_i) - \sum_{i=o}^{n} \delta^i u(c_i^*) < \sum_{i=o}^{n} \delta^i u_i'(c_i^*)(c_i - c_i^*) = \sum_{i=o}^{n} \lambda_i (c_i - c_i^*) =$$

$$= \lambda_o (k_o^{+*} - k_o^+) + \lambda_1 \left(\int_o^1 f(k(t))dt + k_o^+ - k_1^+ - \int_o^1 f(k^*(t))dt - k_o^{+*} + k_1^{+*}\right) + \ldots + \lambda_{n-1}$$

$$\left(\int_{n-2}^{n-1} f(k(t))dt + k_{n-2}^+ - k_{n-1}^+ - \int_{n-2}^{n-1} f(k^*(t))dt - k_{n-2}^{+*} + k_{n-1}^{+*}\right) + \lambda_n \left(\int_{n-1}^{n} f(k(t))dt + k_{n-1}^+\right.$$

$$\left. - \int_{n-1}^{n} f(k^*(t))dt - k_{n-1}^{+*}\right) = \Omega \tag{4}$$

The strict concavity of $\int_i^{i+1} f(k(t))dt$ in $k_i^+$ implies

$$\Omega < \left(-\lambda_o + \lambda_1 \left(\int_o^1 f'(k^*(t)) \frac{\partial k^*}{\partial k_o^{+*}} dt + 1\right)\right)(k_o^+ - k_o^{+*}) + \ldots +$$

$$\left(-\lambda_{n-1} + \lambda_n \left(\int_{n-1}^{n} f'(k^*(t)) \frac{\partial k^*}{\partial k_{n-1}^{+*}} dt + 1\right)\right)(k_{n-1}^+ - k_{n-1}^{+*}) \tag{5}$$

which is zero because of (3). This completes the proof.

Next we turn to the interpretation of the first order conditions. Relations (3) imply

$$\frac{u'(c_i)}{\delta u'(c_{i+1})} = \int_i^{i+1} f'(k) \frac{\partial k}{\partial k_i^+} dt + 1 \tag{6}$$

The right-hand-side of (6) can be interpreted as the marginal rate of transformation. Suppose $k_{i-1}^+$ and $k_{i+1}^+$ are fixed. Now a decrease in $c_i$ by one unit implies an increase in $k_i^+$ by one unit, which in turn implies an increase in $c_{i+1}$ by $\int_i^{i+1} f'(k) \frac{\partial k}{\partial k_i^+} dt + 1$, which, therefore, can be interpreted as the marginal rate of transformation between $c_i$ and $c_{i+1}$. Of cource the left-hand-side of (6) is the marginal rate of substitution.

Finally we show how replacing the unit interval between consumption decisions by $\Delta t$ and allowing $\Delta t$ to tend to zero we obtain through (3) the Eulerian relation of the calculus of variations.

We take $\Delta t = \frac{1}{N}$ where N is a positive integer that will tend to infinity. The production constraints will now be of the form $c_{t+\Delta t}\Delta t + k^+_{t+\Delta t} = \int_t^{t+\Delta t} f(k(t'))dt' + k^+_t$, except for the first one which remains the same and the last one in which $k^+_{t+\Delta t}$ is set equal to zero. Therefore with respect to the marginal rate of transformation, arguing as above, we have for small $\Delta t$

$$\text{MRT} = -\frac{dc_{t+\Delta t}}{dc_t} = \int_t^{t+\Delta t} f'(k(t'))\frac{\partial k}{\partial k^+_t}\,dt' + 1 = f'(k^+_t)\Delta t + 1 \tag{7}$$

where $f'(k^+_t)\Delta t$ is obtained by using the first order approximation of the integral and invoking $\left(\frac{\partial k}{\partial k^+_t}\right)_{t'=t} = 1$.

The question arises as to the limit of the utility function as the decision interval is divided into smaller and smaller intervals of equal length. One can argue as follows. First, the discount factor is replaced by

$$\left(\frac{1}{1+\beta\Delta t}\right) \text{ and } \lim_{\Delta t\to 0}\left(\frac{1}{1+\beta\Delta t}\right)^{t/\Delta t} = \exp(-\beta t).$$

Now we know that in the limit the utility sum is replaced by an integral. In the utility sum in (1) we interpret $\delta^i u(c_i)$ to mean that quantity $c_i$ allocated to consumption at the beginning of the interval implies a constant utility rate, $\delta^i u(c_i)$, throughout the unit length interval. However when the length of the decision interval is changed to $\Delta t$, consumption allocated at the beginning of the interval implies a constant utility rate weighted by $\Delta t$. Taking into account also the new discount factor we write

$$U = \sum_{i=0}^{\frac{n+1}{\Delta t}-1}\left(\frac{1}{1+\beta\Delta t}\right)^i u(c_i)\Delta t \tag{8}$$

where index i and time t are related through $t = i\Delta t$.

Obviously as we have extended consumption decisions into $\left[\frac{n}{\Delta t}, \frac{n+1}{\Delta t} - 1\right]$, we assume that production is also carried out during this interval.

For the modified model, the marginal rate of substitution between $c_t$ and $c_{t+\Delta t}$ is

$$MRS = -\frac{dc_{t+\Delta t}}{dc_t} = (1+\beta\Delta t)\ \frac{u'(c_t)\ \Delta t}{u'(c_{t+\Delta t})\Delta t}, \tag{9}$$

and MRS = MRT is obtained from the corresponding to (3) first order conditions.

Next, u(c) is well behaved in c and applying the mean value theorem on the function $\frac{1}{u'(c)}$ we obtain from (9)

$$1+\beta\Delta t - \frac{u'(c_t)}{u'(\xi)^2}\ u''(\xi)(c_{t+\Delta t}-c_t) = f'(k_t^+)\Delta t + 1 \tag{10}$$

where $\xi$ lies between $c_t$ and $c_{t+\Delta t}$, and the right-hand-side is from (7). The question arises as to whether, for sufficiently small $\Delta t$, we can replace $\xi$ by $c_t$ in (10). Relations (3) imply for bounded $f'(k)$, that $(c_{t+\Delta t}-c_t) \to 0$ as $\Delta t \to 0$ and therefore in this case $\xi$ can be replaced by $c_t$. Hence in the limit we obtain from (10)

$$\beta - \frac{1}{u'(c_t)}\, u''(c_t)\ \frac{dc_t}{dt} = f'(k_t) \ \text{ or } \ \exp(-\beta t)u'(c_t)f'(k_t) + \frac{d}{dt}\exp(-\beta t)u'(c_t) = 0 \tag{11}$$

which is the Eulerian relation of the calculus of variations.

### 3. Consumption at All Points in Time and Production in Discrete Times

Now we consider Model 2

$$\begin{aligned}
&\text{Maximize} && U = \int_o^n \exp(-\beta t)\,u(c(t))\,dt \\
&\text{Subject to} && \int_o^1 c(t)\,dt + k_o = W \\
& && \int_1^2 c(t)\,dt + k_1 = g(k_o) + k_o \\
& && \cdots\cdots\cdots\cdots\cdots\cdots \\
& && \int_{n-1}^n c(t)\,dt = g(k_{n-2}) + k_{n-2} \\
& && c(t),\ k_i \geqslant 0
\end{aligned} \tag{12}$$

where W is a positive constant and $\beta > 0$. The notation is the obvious one and once more the same good can be used for consumption and production purposes. For the production function we assume $g'(k)>0$ and $g''(k)<0$, $g(k)>0$ and for the instantaneous utility function $u'(c)>0$, $u''(c)<0$ and $u'(0)=\infty$. As in Model 1, a positive terminal capital stock requirement could have been accommodated easily, and a free disposal assumption would have made no difference to the solution.

We now have to choose the $k_i$'s and allocate the quantities available for consumption optimally.

In order to characterize the solution we form the Langrangian function

$$V=\int_o^n \exp(-\beta t)\,u(c(t))\,dt+\lambda_o\Big(W-k_o-\int_o^1 c(t)\,dt\Big)+\lambda_1\Big(g(k_o)+k_o-k_1-\int_1^2 c(t)\,dt\Big) \quad (13)$$
$$+\lambda_2\Big(g(k_1)+k_1-k_2-\int_2^3 c(t)\,dt\Big)+\ldots+\lambda_{n-2}\Big(g(k_{n-3})+k_{n-3}-k_{n-2}-\int_{n-2}^{n-1} c(t)\,dt\Big)$$
$$+\lambda_{n-1}\Big(g(k_{n-2})+k_{n-2}-\int_{n-1}^{n} c(t)\,dt\Big)$$

The problem in (12) is a mixture of maximization problems, one with respect to a finite number of variables and one with respect to paths c(t).

First order conditions for a maximum are

$$\exp(-\beta t)\,u'(c(t))=\lambda_o \quad t\varepsilon[0,\,1] \qquad W-k_o-\int_o^1 c(t)\,dt=0$$
$$\exp(-\beta t)\,u'(c(t))=\lambda_1 \quad t\varepsilon[1,\,2] \qquad g(k_o)+k_o-k_1-\int_1^2 c(t)\,dt=0$$
$$\ldots\ldots\ldots\ldots \qquad g(k_1)+k_1-k_2-\int_2^3 c(t)\,dt=0$$
$$\exp(-\beta t)\,u'(c(t))=\lambda_{n-2} \quad t\varepsilon[n-2,\,n-1] \qquad \ldots\ldots\ldots\ldots$$
$$\exp(-\beta t)\,u'(c(t))=\lambda_{n-1} \quad t\varepsilon[n-1,\,n] \qquad g(k_{n-3})+k_{n-3}-k_{n-2}-\int_{n-2}^{n-1} c(t)\,dt=0$$
$$g(k_{n-2})+k_{n-2}-\int_{n-1}^{n} c(t)\,dt=0 \quad (14)$$

$$-\lambda_o+\lambda_1(g'(k_o)+1)=0$$
$$-\lambda_1+\lambda_2(g'(k_1)+1)=0$$
$$\ldots\ldots\ldots\ldots$$
$$-\lambda_{n-3}+\lambda_{n-2}(g'(k_{n-3})+1)=0$$
$$-\lambda_{n-2}+\lambda_{n-1}(g'(k_{n-2})+1)=0$$

Next we consider the global optimality of the above conditions.

Proposition. If path $c^*(t)$ and vector $(k_o^*, k_1^*,\ldots,k_{n-2}^*)$ satisfy (14) for some $(\lambda_o, \lambda_1,\ldots, \lambda_{n-1})$ then they are the unique globally optimal consumption path and capital stock vector.

Proof. All λ's will be positive. Consider any other feasible path $c(t)$ and vector $(k_o, k_1,\ldots, k_{n-2})$ such that $c(t)$ differs from $c^*(t)$ over some interval. The strict concavity of $u(c)$ implies

$$\int_o^n \exp(-\beta t)\,u(c(t))\,dt - \int_o^n \exp(-\beta t)\,u(c^*(t))\,dt < \int_o^n \exp(-\beta t)\,u'(c^*(t))(c(t)-c^*(t))\,dt = \Omega \quad (15)$$

Now because of the first set of relations in the first order conditions, (14),

$$\Omega = \int_o^1 \lambda_o(c(t)-c^*(t))\,dt + \int_1^2 \lambda_1(c(t)-c^*(t))\,dt + \int_2^3 \lambda_2(c(t)-c^*(t))\,dt + \ldots$$
$$+\int_{n-2}^{n-1} \lambda_{n-2}(c(t)-c^*(t))\,dt + \int_{n-1}^{n} \lambda_{n-1}(c(t)-c^*(t))\,dt, \quad (16)$$

and invoking the second set of these relations we have

$$\Omega = \lambda_o(-k_o+k_o^*) + \lambda_1(g(k_o)+k_o-k_1-g(k_o^*)-k_o^*+k_1^*) +$$
$$\lambda_2(g(k_1)+k_1-k_2-g(k_1^*)-k_1^*+k_2^*) + \ldots + \lambda_{n-2}(g(k_{n-3})+k_{n-3}-k_{n-2}-$$
$$-g(k_{n-3}^*)-k_{n-3}^*+k_{n-2}^*) + \lambda_{n-1}(g(k_{n-2})+k_{n-2}-g(k_{n-2}^*)-k_{n-2}^*) \quad (17)$$

Next invoking the strict concavity of $g(k)$ we obtain

$$\Omega < (k_o-k_o^*)(-\lambda_o+\lambda_1(g'(k_o^*)+1)) + (k_1-k_1^*)(-\lambda_1+\lambda_2(g'(k_1^*)+1)) + \ldots$$
$$+(k_{n-2}-k_{n-2}^*)(-\lambda_{n-2}+\lambda_{n-1}(g'(k_{n-2}^*)+1)) \quad (18)$$

which is zero because of the last set of relations in (14). This completes the proof.

Below we turn to the interpretation of the first order conditions. First we notice that in order to obtain the solution of Model 2 we could have argued as follows.

Suppose $k_o ,\ldots, k_{n-2}$ were fixed. This would mean that the total quantity available for consumption during any period was fixed. The optimal allocation of this quantity within the period would be obtained by solving

Maximize $\quad U_i = \int_i^{i+1} \exp(-\beta t)\,u(c(t))\,dt$

Subject to $\int_i^{i+1} c(t)\,dt = A_i$

$c(t) \geqslant 0$

where $A_i$ denotes the total quantity allocated for consumption in $[i, i+1]$ 19

The solution of (19) will be a path $c(t)$, $t\varepsilon[i, i+1]$, which will depend on $A_i$, which in turn depends on the relevant $k_i$'s. For the full solution of Model 2, it remains to select $k_o, k_1, \ldots, k_{n-2}$.

The first two sets of relations in (14) solve all consumption allocation problems for fixed $k_o, k_1, \ldots, k_{n-2}$. Then given the first set of relations, the last two sets in (14) determine the optimal values of $k_o, k_1, \ldots, k_{n-2}$.

Now within any interval $[i, i+1]$ the first set of relations in (14) imply

$$\frac{\exp(-\beta t)u'(c(t))}{\exp(-\beta t')u'(c(t'))} = 1 \qquad t, t'\varepsilon[i, i+1] \tag{20}$$

The left-hand-side of (20) can be recognized as the marginal rate of substitution between $c(t)$ and $c(t')$. The right-hand-side is the marginal rate of transformation between $c(t)$ and $c(t')$. A fixed quantity has been allocated to consumption and any one unit can simply be transferred between any two points in time.

Next from the first and last set of relations in (14) we obtain

$$\frac{\exp(-\beta i)u'(c(i))}{\exp(-\beta(i+1))u'(c(i+1))} = g'(k_i)+1 \qquad i=0, 1, \ldots, n-2 \tag{21}$$

Relation (21) expresses the equality of the marginal rate of substitution to the marginal rate of transformation between adjacent periods.

Let $C_i = \int_i^{i+1} c(t)\,dt$, $i=0, 1, \ldots, n-1$, the total quantity available for consumption in $[i, i+1]$. Suppose $k_{i-1}$ and $k_{i+1}$ are fixed. A decrease in $C_i$ by one unit implies an increase in $k_i$ by one unit which in turn implies an increase in $C_{i+1}$ by $g'(k_i)+1$. Therefore

$$MRT = -\frac{dC_{i+1}}{dC_i} = g'(k_i)+1 \tag{22}$$

On the other hand, for a constant utility level variations in $C_i$ and $C_i+1$ must satisfy

$$\int_{i}^{i+1}\exp(-\beta t)\,u'(c(t))\delta c(t)\,dt+\int_{i+1}^{i+2}\exp(-\beta t)\,u'(c(t))\,\delta c(t)\,dt=0 \qquad (23)$$

which in view of the first set of relations in (14) implies

$$u'(c(i))\int_{i}^{i+1}\delta c(t)\,dt+\exp(-\beta)\,u'(c(i+1))\int_{i+1}^{i+2}\delta c(t)\,dt=0 \qquad (24)$$

and since $dC_i=\int_{i}^{i+1}\delta c(t)\,dt$ the left-hand-side of (21) can be recognized as the marginal rate of substitution between $C_i$ and $C_{i+1}$.

Next we show that as the production period becomes infinitesimally small relation (21) tends to the Eulerian relation of the calculus of variations. If the production period is of length $\Delta t$ relation (21) is replaced by

$$\frac{u'(c(t))}{\exp(-\beta\Delta t)\,u'(c(t+\Delta t))}=g'(k_t)\Delta t+1 \qquad (25)$$

The left-hand-side is obvious. The right-hand-side follows from the fact that if $k_t$ produces $g(k_t)$ when the production period is of unit length then, approximately, $k_t$ produces $g(k_t)\Delta t$ when the production period is of length $\Delta t$.

Now irrespective of the fixed length of intervals, $\Delta t$, $c(t)$ is not necessarily continuous in time, in particular as we cross from one interval to the next. On the other hand, $u(c)$ is well behaved in c and applying the mean value theorem on the function $\frac{1}{u'(c)}$ we can write for some value $\xi$ between $c(t)$ and $c(t+\Delta t)$

$$\frac{u'(c(t))}{u'(c(t+\Delta t))}=1-\frac{u'(c(t))}{u'(\xi)^2}\,u''(\xi)\,(c(t+\Delta t)-c(t)) \qquad (26)$$

The question arises as to whether, for sufficiently small $\Delta t$, we can replace $\xi$ by $c(t)$ in (26). Relation (25) implies, for bounded $g'(k_t)$, that $(c(t+\Delta t)-c(t))\rightarrow 0$ as $\Delta t\rightarrow 0$ and therefore in this case $\xi$ can be replaced by $c(t)$. Approximating also $\exp(\beta\Delta t)$ by $1+\beta\Delta t$ we can replace (25) by

$$1+\beta\Delta t-\frac{1}{u'(c(t))}\,u''(c(t))\,(c(t+\Delta t)-c(t))=g'(k_t)\Delta t\mp 1 \qquad (27)$$

which implies

$$\beta-\frac{1}{u'(c(t))}\,u''(c(t))\,\frac{dc_t}{dt}=g'(k_t) \qquad \text{or}$$

$$\exp(-\beta t)u'(c(t))g'(k_t) + \frac{d}{dt}\exp(-\beta t)u'(c(t)) = 0 \tag{28}$$

the Eulerian relation of the calculus of variations.

## 4. Conclusion

We have discussed, for discrete – continuous time models, global optimality conditions and obtained their interpretation. It was also shown how, as the distance between the equally spaced discrete times tends to zero, the optimality conditions tend to the Euler equation of the calculus of variations. Special aspects of the two models discussed are the following.

In Model 1, where consumption decisions are taken at discrete times, we are seeking the solution of a maximization problem with a finite number of variables. An interesting feature of the discussion is how the strict concavity of the instantaneous production function is used in the sufficiency argument. Also of special interest is the manner in which the utility sum is assumed to adjust as the distance between decision times becomes shorter. It implies that in the limit the first order conditions are reduced to the Eulerian relation of the continuous model.

In Model 2, where consumption takes place at all points in time, there are in effect two problems. One is the optimal allocation of fixed quantities over given intervals and the other is the optimal choice of capital stocks at the discrete times. Equality between marginal rate of substitution and marginal rate of transformation holds now within and between periods. Continuity of the consumption rate path as we cross intervals is not guaranteed, however it is again possible to obtain in the limit the characteristic equation of the calculus of variations.

## Footnotes

1. References reflecting the basic literature are given at the end of the paper. Burmeister and Dobell (1970), and Takayama (1985) present a very good discussion of the main issues.

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