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ON THE DERIVATION AND SOLUTION OF THE BLACK-SCHOLES OPTION PRICING MODEL: A STEP-BY-STEP GUIDE

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Abstract

The derivation and solution of the celebrated Black-Scholes Option Pricing Formula is set out in rather more detail than has appeared in the literature so far. One problem with the Black-Scholes analysis is that the mathematical skills required in the derivation and particularly in the solution of the model are fairly advanced and probably unfamiliar to most economists. This paper derives the partial differential equation for the call option price and gives *full* details of its solution. All the necessary mathematics are given in three appendices. It is anticipated that the mathematical methods detailed here will be of wider applicability in Economics and Finance. (JEL G13).

1. Introduction

Interest in the theory of option pricing received a major stimulus in 1973 with the publication of a pioneering paper by Black and Scholes (2). The Black-Scholes paper represents a milestone in the option pricing literature for several reasons: on the one hand it was the first realistic general equilibrium model of option pricing; but it is equally important in the sense that it has engendered much subsequent literature on the valuation of many types of contingent claim. (See, for example, references (3), (8), and (13) in the bibliography). Finally, it had important implications for empirical work. Since the price of a call option depends on only five quantities (all of which are either directly observable or easily measurable) the empirical calculation of option prices can be a relatively simple task.

One problem with the Black-Scholes analysis, however, is that the mathematical skills required in the derivation and solution of the model are fairly advanced and probably unfamiliar to many economists. The analysis is essentially in two parts. Firstly, Black-Scholes show how a riskless hedge can be constructed when the stochastic process for the underlying asset price is lognormal. It is thus shown that the call option price is determined by a certain second order partial differential equation. The second part of the analysis involves solving this partial differential equation and thus deriving the celebrated Black-Scholes pricing formula.

The first part of the analysis has received considerable attention in the literature. It has been discussed in detail by Cox and Ross (8), Chow (6), Cox and Rubinstein (9), Smith (16) and Malliaris (11) as well as in the original Black-Scholes paper - and this is a by no means exhaustive list of references. The second part is rather more difficult and details of the solution technique for the partial differential equation of the option price have not been given coverage in the literature. Black and Scholes give few details and simply state the solution —the Black-Scholes formula. Several authors have searched for alternative ways of deriving the Black-Scholes formula that avoid the necessity of solving a partial differential equation. Usually these assume making certain assumptions about investor behaviour and the work of Cox and Rubinstein (9) is typical of this type of approach.

The purpose of this paper is to set out the derivation and Solution of the Black-Scholes model in some detail. The following section deals with the derivation of the partial differential equation for the option price, section three covers the solution of the partial differential equation and section four offers some concluding remarks. The material in section two has already received much attention in the literature and is thus dealt with rather briefly here. In contrast, the material of section three has received little (if any!) attention and is discussed here in some detail.

Mathematical appendices are provided for those areas of mathematics which are likely to be unfamiliar to many economists. Our aim is to make the type of analysis required in the Black-Scholes model more accessible and thus to stimulate further interest in the theory of pricing of contingent claims.

2. The Black-Scholes model

The holder of a European call option has the right to purchase a unit of a given stock at a certain price, the exercise price, on a certain date, the exercise date. Assume that:

$$P(t) = F(t, S(t)) \quad (1)$$

where P(t) is the price of a call option at time t and S(t) is the stock price at time t. The stockprice is allowed to vary continuously and is assumed to be generated by the following stochastic differential equation (see appendix A):

$$dS = \alpha S dt + \sigma S dz \quad (2)$$

where a and σ are positive constants and dz is the increment of a Wiener process (see appendix A). Thus:

$$dz \sim N(O; dt) \quad (3)$$

Consider an investor who builds a portfolio of three assets; the stock, an option on the stock and a riskless asset such as a Government bond. This latter earns the riskless competitive rate of return, r, which is assumed to be constant here for simplicity. Assume that there are no transactions costs, the market operates continuously and no dividends or other distributions are paid to stock holders. Let:

$N_1(t)$ = number of units of the stock held at time t

$N_2(t)$ = number of units of the stock on which an options is held at time t

and let Q(t) be the number of units of currency invested in the riskless asset at time t. Then the nominal value of the portfolio at time t is:

$$\Pi(t) = N_1(t).S(t) + N_2(t)P(t) + Q(t) \quad (4)$$

Differentiating (1) using Ito's lemma (see appendix A) gives:

$$dP = F_t dt + F_s dS + \tfrac{1}{2}F_{ss}(dS)^2 \quad (5)$$

But, from (2) and (3)

$$(dS)^2 = \sigma^2 S^2 dt \quad (6)$$

Therefore, substituting from (2) and (6) into (5) gives:

$$dP = F_t dt + \alpha S F_s dt + \sigma S F_s dz + \tfrac{1}{2}\sigma^2 S^2 F_{ss} dt$$

or rearranging:

$$dP = [F_t + \alpha S F_s + \tfrac{1}{2}\sigma^2 S^2 F_{ss}]\, dt + \sigma S F_s dz \quad (7)$$

Changes in the nominal value of the portfolio arise because of changes in the *prices* of the assets because at a point in time the *quantities* are given, i.e. $dN_1(t) = dN_2(t) = 0$. Therefore:

$$d\Pi = N_1 dS + N_2 dP + dQ \tag{8}$$

But by definition:

$$dQ = rQdt \tag{9}$$

So, substituting from (2), (7) and (9) into (8) gives:

$$\begin{aligned} d\Pi &= N_1 [\alpha S dt + \sigma S Dz] + rQdt \\ &+ N_2 \{ [F_t + \alpha S F_s + \tfrac{1}{2}\sigma^2 S^2 F_{ss}] dt + \sigma S F_s dz \} \end{aligned}$$

or, rearranging:

$$\begin{aligned} d\Pi &= \{ N_1 \alpha S + rQ + N_2 [F_t + \alpha S F_s + \tfrac{1}{2}\sigma^2 S^2 F_{ss}] \} dt \\ &+ [N_1 + N_2 F_s] \sigma S dz \end{aligned} \tag{10}$$

For arbitrary quantities of the three assets, the change in portfolio value is stochastic (via the last term in equation (10)).

But if the quantities of stock and call are chosen such that $N_1+N_2F_s = 0$ then the portfolio becomes riskless and thus earns the competitive riskless rate of return, i.e. we let:

$$N_1 = -N_2 F_s \tag{11}$$

and then:

$$\frac{d\Pi}{\Pi} = r\, dt \tag{12}$$

Substituting (1) and (11) into (4) and substituting (11) into (10) gives, respectively:

$$\Pi = N_2 [F - SF_s] + Q \tag{13}$$

$$d\Pi = \{ N_2 [F_t + \tfrac{1}{2}\sigma^2 S^2 F_{ss}] + rQ \} dt \tag{14}$$

Finally, substitute (13) and (14) into (12) and:

$$\frac{\{ N_2 [F_t + \tfrac{1}{2}\sigma^2 S^2 F_{ss}] + rQ \} dt}{N_2 [F - SF_s] + Q} = r\, dt$$

Simplifying and rearranging gives the following second order partial differential equation in the option price:

$$F_t + rSF_s + \frac{1}{2}\sigma^2 S^2 F_{ss} - rF = 0 \tag{15}$$

3. Solution of the Partial Differential Equation

Let t* denote the exercise date and let E denote the exercise price. Then we require the solution of equation (15) that also satisfies the boundary condition (see appendix B for the solution of partial differential equations):

$$F(S, t^*) = \max(O; S(t^*) - E) \tag{16}$$

Equation (15) is a linear second-order partial differential equation of the parabolic type. By choice of suitable coordinates, any parabolic equation can be reduced to the canonical form (see appendix B):

$$G_{xx} = G_y \tag{17}$$

We proceed by letting:

$$F(S, t) = e^{-rT} G(x, y) \tag{18}$$

where $x = x(S, t)$, $y = y(S, t)$ and $T = t^* - t$ (the time to go to the exercise date). Partially differentiating (18) gives:

$$F_s = e^{-rT}[G_x x_s + G_y y_s] \tag{19}$$

$$F_t = re^{-rT} G + e^{-rT}[G_x x_t + G_y y_t] \tag{20}$$

$$F_{ss} = e^{-rT}[G_{xx} x_s^2 + G_{xy} x_s y_s + G_x x_{ss} + G_{yx} x_s y_s + G_{yy} y_s^2 + G_y y_{ss}] \tag{21}$$

Substituting (18) - (21) into (15) gives (after a bit or rearrangement):

$$\frac{1}{2}\sigma^2 S^2 x_s^2 G_{xx} + \sigma^2 S^2 x_s y_s G_{xy} + \frac{1}{2}\sigma^2 S^2 y_s^2 G_{yy}$$

$$+ [\tfrac{1}{2}\sigma^2 S^2 x_{ss} + rSx_s + x_t]\, G_x + [\tfrac{1}{2}\sigma^2 S^2 y_{ss} + rSy_s + y_t]\, G_y = 0 \tag{22}$$

If (22) is to reduce to an equation of the form given in (17) we need the functions x(S, t) and y (S, t) to satisfy the following:

$$\left.\begin{array}{l} \tfrac{1}{2}\sigma^2 S^2 x^2_s + \tfrac{1}{2}\sigma^2 S^2 y_{ss} + rSy_s + y_t = 0 \\ \sigma^2 S^2 x_s y_s = 0 \\ \tfrac{1}{2}\sigma^2 S^2 y^2_s = 0 \\ \tfrac{1}{2}\sigma^2 S^2 x_{ss} + rSx_s + x_t = 0 \end{array}\right) \quad (23)$$

The third of these conditions shows that y does **not** depend upon S. Thus it is a function of t only. Using this, conditions (23) reduce to:

$$\left.\begin{array}{l} \tfrac{1}{2}\sigma^2 S^2 x^2_s + y_t = 0 \\ \tfrac{1}{2}\sigma^2 S^2 x_{ss} + rSx_s + x_t = 0 \end{array}\right) \quad (24)$$

Since y is a function only of t it is clear from the first of conditions (24) that Sx_s must not depend upon S. Looking for the simplest possible transformation, let:

$$x(S, t) = \beta \ell n(S) + \gamma T \quad (25)$$

where β and γ are constants. Differentiating equation (25) and substituting into conditions (24) gives:

$$\left.\begin{array}{l} \tfrac{1}{2}\sigma^2 \beta^2 + y_t = 0 \\ -\tfrac{1}{2}\sigma^2 \beta + r\beta - \gamma = 0 \end{array}\right) \quad (26)$$

So if we let $\beta = 1$, $\gamma = r - \tfrac{1}{2}\sigma^2$ and $y(t) = \tfrac{1}{2}\sigma^2 T$ the simplest transformation to give the desired result is:

$$x(S, t) = \ell n(S) + (r - \tfrac{1}{2}\sigma^2)T \quad (27)$$

$$y(t) = \tfrac{1}{2}\sigma^2 T \quad (28)$$

It is worth noting that this is a much simpler transformation than that suggested by Black-Scholes.

Applying (27) and (28) to (15) and (16) thus gives

$$G_{xx} = G_y \quad (17)$$

with boundary condition:

$$f(x) \equiv G(x, O) = \max [O; e^x - E] \tag{29}$$

To solve this equation assume that G is separable and may be written:

$$G(x, y) = X(x)\, Y(y) \tag{30}$$

Then $G_{xx} = X''(x)\, Y(y)$ and $G_y = X(x)\, Y'(y)$

$$X''(x)\, Y(y) = X(x)\, Y'(y) \tag{31}$$

Divide each side of (31) by G and thus:

$$\frac{X''}{X} = \frac{Y'}{Y} = k \tag{32}$$

where k is the separation constant. This suggests the following pair of *ordinary* differential equations:

$$X'' - kX = 0 \tag{33}$$

$$Y' - kY = 0 \tag{34}$$

For (33), since k can assume **any** value, the most general solution is:

$$X(x) = Ae^{i\lambda x} \tag{35}$$

where A is an arbitrary constant and $\lambda = \sqrt{k}$; i is the imaginary unit, defined by $i \equiv \sqrt{-1}$.

For (34), the most general solution is:

$$Y(y) = Be^{-\lambda^2 y} \tag{36}$$

where B is a constant. Thus the most general product solutions (referring to (30)) are:

$$G(x, y) = ce^{[i\lambda x - \lambda^2 y]} \tag{37}$$

where c (=AB) is an arbitrary constant.

Since λ can take **any** value we apply the superposition principle to obtain the most general solution:

$$G(x, y) = \int_{-\infty}^{\infty} c(\lambda) e^{[i\lambda x - \lambda^2 y]}\, d\lambda \tag{38}$$

When y=0, the solution must satisfy the boundary condition $G(x, 0) \equiv f(x)$ given in (29). Thus setting y=0 in (38):

$$f(x) = \int_{-\infty}^{\infty} c(\lambda)e^{i\lambda x}d\lambda \tag{39}$$

From (39) we recognise that f(x) and c(λ) are a Fourier transform pair (see appendix C). Therefore:

$$c(\lambda) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx \tag{40}$$

Substituting (40) into (39):

$$G(x, y) = \frac{1}{2\Pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-[i(u-x)\lambda + \lambda^2 y]}\, du.d\lambda. \tag{41}$$

Reversing the order of integration and integrating first with respect to λ:

$$G(x, y) = \frac{1}{2\sqrt{\Pi y}}\int_{-\infty}^{\infty} f(u)\, e^{-(u-x)^2/4y}\, du \tag{42}$$

Substituting for f (u) from (29) gives

$$G(x, y) = \frac{1}{2\sqrt{\Pi y}}\int_{\ell n(E)}^{\infty} (e^u - E)\, e^{-(u-x)^2/4y}\, du$$

Next, using (18), (27) and (28) gives an expression for F (S, t), i.e.

$$F(S, t) = \frac{e^{-rT}}{\sigma\sqrt{2\Pi T}}\int_{\ell n(E)}^{\infty} (e^u - E)\, e^{-[u - \ell n(S) - (r - \frac{1}{2}\sigma^2)T]^2/2\sigma^2 T}\, du \tag{43}$$

To evaluate this integral first write it as the pair of integrals:

$$F(S, t) = \frac{e^{-rT}}{\sigma\sqrt{2\Pi T}}\int_{\ell n(E)}^{\infty} e^{u - [u - \ell n(s) - (r - \frac{1}{2}\sigma^2)T]^2/2\sigma^2 T}\, du$$

$$- \frac{Ee^{-rT}}{\sigma\sqrt{2\Pi T}}\int_{\ell n(E)}^{\infty} e^{-[u - \ell n(s) - (r - \frac{1}{2}\sigma^2)T]^2/2\sigma^2 T}\, du \tag{44}$$

In the first integral on the right-hand side of equation (44) add and subtract the quantity [$\ell n(S) + rT$] to the exponent under the integral sign. This integral is then transformed into:

$$\frac{e^{-rT}}{\sigma\sqrt{2\Pi T}}\int_{\ell n(E)}^{\infty} e^{u-[u-\ell n(S)-(r-\frac{1}{2}\sigma^2)T]^2/2\sigma^2 T}\,du = \frac{S}{\sigma\sqrt{2\Pi T}}\int_{\ell n(E)}^{\infty} e^{-[u-\ell n(S)-(r+\frac{1}{2}\sigma^2)T]^2/2\sigma^2 T}\,du$$

Now let $\Omega = \dfrac{u - \ell n(S) - (r + \frac{1}{2}\sigma^2)\,T}{\sigma\sqrt{T}}$

Then this integral is further transformed and

$$\frac{e^{-rT}}{\sigma\sqrt{2\Pi T}}\int_{\ell n(E)}^{\infty} e^{u-[u-\ell n(S)-(r-\frac{1}{2}\sigma^2)T]^2/2\sigma^2 T}\,du = \frac{S}{\sqrt{2\Pi}}\int_{-\infty}^{[\ell n(S/E)+(r+\frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}} e^{-\Omega^2/2}\,d\Omega$$

$$= S\ \Phi\left(\frac{\ell n\,(S/E) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \qquad (45)$$

where Φ (·) is the cumulative distribution function for the standard normal variate.

In the same way, for the second integral on the right-hand side of (44) let:

$$w = \frac{u - \ell n(S) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Then:

$$-\frac{Ee^{-rT}}{\sigma\sqrt{2\Pi T}}\int_{\ell n(E)}^{\infty} e^{-[u-\ell n(S)-(r-\frac{1}{2}\sigma^2)T]^2/2\sigma^2 T}\,du = -\frac{Ee^{-rT}}{\sqrt{2\Pi}}\int_{-\infty}^{[\ell n(S/E)+(r+\frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}} e^{-w^2/2}\,dw$$

$$= Ee^{-rT}\ \Phi\left(\frac{\ell n\,(S/E) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \qquad (46)$$

Finally, substituting (45) and (46) into (44) gives the Black-Scholes formula for the call-option price:

$$F(S, t) = S\,\Phi\left(\frac{\ln(S/E) + (r + \tfrac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - Ee^{-rT}\,\Phi\left(\frac{\ln(S/E) + (r - \tfrac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \quad (47)$$

4. Concluding Remarks

Equation (47) is the formula derived by Black and Scholes and subsequently much repeated elsewhere in the literature. As previously remarked it depends on only five quantities; the stock price, exercise price and time to maturity are all directly observable. For the riskless rate of return one could use as a proxy the Treasury Bill rate or the London Inter-Bank Offer rate, suitably adjusted so as to provide an instantaneous rather than an annual rate. For the variance rate, σ, various possibilities exist for its estimation. Latane and Rendelman (10) suggest an 'implied variance rate' and Cox and Rubinstein (8) show how it may be estimated directly from stock price data. Recently Chappell and Chrystal (5) have derived an unbiased estimate of σ using the logarithms of stock price and adjusting for non-linearity.

Appendix A. Stochastic Differential Equations

Only a bare outline of the fundamental concepts used in the paper can be given here. The reader is invited to refer to references (1), (2) or (12) for fuller details. Useful summaries are also to be found in (6) and (11).

Consider the stochastic *difference* equation:

$$x(t + h) - x(t) = hAx(t) + \upsilon(t + h) - \upsilon(t) \quad (A1)$$

where A is constant and $\upsilon(t + h) - \upsilon(t)$ is a random term with zero mean. Successive increments are assumed to be statistically independent through time, whatever the choice of h. For $h = 1$, let the variance of $\upsilon(t+h) - \upsilon(t)$ be σ^2. For smaller values of h divide the unit time interval from t to t+1 into n increments of equal length. Since the increments $\upsilon(t+h) - \upsilon(t)$ are statistically independent and $h = 1/n$ it follows that the variance of $\upsilon(t+1) - \upsilon(t)$ is n times the variance of $\upsilon(t+h) - \upsilon(t)$: But we know that the variance of $\upsilon(t+1) - \upsilon(t)$ is σ^2; so it follows that the variance of $\upsilon(t+h) - \upsilon(t)$ is $\sigma^2/n = h\sigma^2$. This is most important and is the fundamental point of difference between 'normal' calculus and stochastic calculus; terms involving the square of $\upsilon(t+h) - \upsilon(t)$ are of order h and not h^2. As h becomes very small, let $h = dt$ and then:

$$dx = Axdt + d\upsilon \quad (A2)$$

where $E(d\upsilon) = 0$ and $Var(d\upsilon) = \sigma^2 dt$. Equation (A2) is a *stochastic differential equation.* Note that we cannot divide through by dt and take the limit as dt tends to zero since the variance of $d\upsilon$ becomes infinite in the limit. If we assume that the successive increments, $d\upsilon$, are normally distributed then the stochastic process is known are a *Wiener process* (or Brownian motion). In the engineering literature the quantity $d\upsilon/dt$ is sometimes referred to as continuous time white noise, but this need not concern us here. See, for example, reference (1) for further details.

Itô's Lemma. Only a simplified version of this important result is given here. For a more general treatment (extension to vector-valued stochastic processes etc.) see reference (1).

Consider the non-linear stochastic differential equation:

$$dx = f(x, t)\, dt + \sigma(x, t)\, dz \qquad \text{(A3)}$$

where dz is a Wiener process with zero mean and unit variance parameter. i.e. $dz \sim N(O, dt)$. Let $y= g(x, t)$ be continuously differentiable in t and twice continuously differentiable in x. We now derive a stochastic differential equation in y. Let x satisfy (A3) then:

$$dy = y_t dt + y_x dx + \tfrac{1}{2}\, y_{xx} dx^2 + o(dt) \qquad \text{(A4)}$$

where o(dt) signifies the sum of terms of smaller order than dt which can thus be "disregarded". But dx is given by (A3) and

$$\begin{aligned} dx^2 &= [\, f(x, t)dt + \sigma(x, t)dz\,]^2 \\ &= \sigma^2(x, t)dz^2 + o(dt) \\ &= \sigma^2(x, t)dt + o(dt) \end{aligned} \qquad \text{(A5)}$$

Therefore, substituting (A3) and (A5) into (A4) gives the following stochastic differential equation for y:

$$dy = [y_t + y_x f(x, t) + \tfrac{1}{2}\, y_{xx}\sigma^2(x, t)]dt + y_x\sigma(x, t)dz \qquad \text{(A6)}$$

Equation (A6) is a simplified version of Ito's Lemma.

APPENDIX B. Partial Differential Equations

Only a brief outline of some of the more important concepts can be given here. The interested reader is invited to consult Churchill and Brown (7) or Nicolescu and Stoka (14); Queen (15) also provides a useful introduction to the subject.

The general linear second order homogeneous partial differential equation in two variables has the form:

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = 0 \qquad \text{(B1)}$$

where A, B, ... F may depend on x and y but not on U(x, y). The equation is often classified as elliptic, hyperbolic or parabolic according to whether B^2-4AC is less than, greater than or equal to zero, respectively. By choice of suitable coordinates any of these equations can be transformed into its *canonical form.* These canonical forms are as follows:

For a hyperbolic equation: $W_{ss} - W_{rr} = 0$

For a parabolic equation: $W_{ss} - W_r = 0$

For an elliptical equation: $W_{ss} + W_{rr} = 0$

where s = s(x, y), r= r(x, y) and W(r, s) = U(x, y).

Once the equation is in its canonical form it is usual to assume that the solution is *separable.* i.e. that:

$$W(r, s) = G(r) . H(s) \qquad \text{(B2)}$$

Differentiating (B2) gives $W_r = G'H$, $W_s = GH'$, $W_{rr} = G''.H$ and $W_{ss} = G.H''$.

Dividing by W (= G.H) gives:

(a) For the hyperbolic equation:

$$\frac{H''}{H} - \frac{G''}{G} = 0$$

This suggests the following system of *ordinary differential equations:*

$$H'' - kH = 0; \quad G'' - kG = 0 \qquad \text{(B3)}$$

where k is a *separation constant.* Similar reasoning results in:

(b) For the parabolic equation:

$$H'' - kH = 0; \quad G' - kG = 0 \tag{B4}$$

(c) For the elliptical eqaution:

$$H'' - kH = 0; \quad G'' + kG = 0 \tag{B5}$$

The way in which one proceeds further largely depends on the boundary conditions that the solution must satisfy and few general principles can be given; frequently Fourier series or Fourier integrals are needed. The following is of fundamental importance however:

Superposition Principle:

Suppose $W_1, W_2, ..., W_N$ are solutions to the linear homogeneous partial differential equation corresponding to those values of the separation constant, k, which also satisfy the boundary conditions. Then $C_1W_1 + C_2W_2 + ... + C_NW_N$, where $C_1, C_2, ..., C_N$ are constants is also a solution. If the separation constant can take **any** value, then the most general solution is:

$$\int_{-\infty}^{\infty} G(r, k) \,.\, H(s, k)\, dk \tag{B6}$$

Appendix C. The Fouruer Transform

The Fourier transform of a function F(y) is defined by:

$$\tilde{F}(x) = \frac{1}{2\Pi} \int_{-\infty}^{\infty} F(y)e^{ixy}dy \tag{C1}$$

The inverse of the Fourier transform is defined by:

$$F(y) = \int_{-\infty}^{\infty} \tilde{F}(x)e^{-ixy}\,dx \tag{C2}$$

$\tilde{F}(x)$ and F(y) are known as a Fourier transform pair.

By Fourier's integral theorem, the integral representation of F(y) is derived by substituting (C1) into (C2), i.e.:

$$F(y) = \frac{1}{2\Pi} \int_{-\infty}^{\infty} e^{-ixy}dx \int_{-\infty}^{\infty} F(u)e^{ixu}du \tag{C3}$$

Sufficient conditions under which Fourier's integral theorem holds are:

(i) F(y) and F′(y) are piecewise continuous in every finite interval $-L<y<L$.

(ii) $\int_{-\infty}^{\infty} |F(y)|\, dy$ converges

(iii) F(y) is replaced by the arithmetic mean of its right-hand and left-hand limits at any points of discontinuity of F(y)

N. B. There is no universal convention as to which integral is the transform and which is its inverse, but ths is really of no importance. Also the position of the factor 2Π varies from author to author -an alternative is to place the factor $1/\sqrt{2\Pi}$ in front of each integral; but again, this is of no importance.

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